

Spontaneous CP violation from vacuum alignment in S_4 models of leptons

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Abstract

We construct models of leptons based on S_4 family symmetry combined with a generalised CP symmetry H_{CP} . We show how the flavon potential can spontaneously break the symmetry $S_4 \rtimes H_{CP}$ down to $Z_2 \times H_{CP}^\nu$ in the neutrino sector, where the choice of preserved CP symmetry H_{CP}^ν is controlled by free (real) parameters in the flavon potential. We propose two realistic models of this kind, one at the effective level and one at the renormalisable level. Both models predict trimaximal lepton mixing with CP being either fully preserved or maximally broken, with the intermediate possibility forbidden by the structure of the models.

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1 Introduction

Recent hints [1–3] for a nonzero reactor mixing angle θ_{13} , and its subsequent measurement by the Daya Bay [4], RENO [5] and Double Chooz [6] Collaborations have revealed an unexpectedly large mixing angle of about 9° to an accuracy comparable to the other two lepton mixing angles, i.e. the solar and atmospheric angles [7]. This observation has had a major impact on models of leptons based on discrete family symmetry (for a recent review see e.g. [8]). On the experimental side, this remarkable discovery opens up the possibility of unravelling (some of) the remaining unknown parameters of the lepton sector, the neutrino mass ordering as well as the leptonic CP violating phases,¹ which is the primary goal of next generation neutrino oscillation experiments.

Concerning the Dirac CP violating phase δ_{CP} , the latest global analyses of the available neutrino oscillation data do not allow to pin down a preferred value at the 3σ level [9–11]. Yet, as we move into the age of precision measurement of the lepton mixing parameters, this state of affairs is bound to change. Inspired by the successes of discrete family symmetries in predicting the lepton mixing angles, it is not unreasonable to believe that the symmetry approach can also lead to predictions for δ_{CP} . To find a realisation of this idea, it appears natural to combine a family symmetry and generalised CP symmetries such that both the mixing angles and the CP violating phases are predicted simultaneously. In this setup, the breaking of the family symmetry would give rise to particular mixing angles, while the breaking of the generalised CP symmetries leads to particular values of the CP phases.

The concept of generalised CP transformations has been around for decades, and it was realised early on that there exists a one-to-one correspondence between the generalised CP transformations and the automorphism group of an imposed family symmetry [12–16]. Recently, the consistent generalised CP transformations (i.e. the automorphism group) of the discrete family symmetry groups of order smaller than 31, with irreducible triplet representations, have been discussed in [17].

Despite these fundamental studies, there has been only little work on constructing explicit models which realise the idea of combining family and generalised CP symmetries [18–20]. In Ref. [19], an S_4 model with imposed generalised CP symmetry is presented which, adopting the type II seesaw mechanism, predicts a normal neutrino mass ordering, a maximal atmospheric angle and a maximal Dirac CP violating phase, $\delta_{CP} = \pm\pi/2$. Based on pure symmetry arguments, a systematic classification of an S_4 family symmetry combined with generalised CP symmetries is performed in [20]. Although the resulting phenomenology was analysed in great detail, a dynamical model which implemented these ideas was not given in [20].

In this paper we shall address the question of breaking a generalised CP symmetry spontaneously in the framework of models based on S_4 family symmetry. We shall construct models of leptons imposing both S_4 family symmetry and the corresponding generalised CP symmetry H_{CP} . We shall then show how the flavon potential can sponta-

¹The leptonic CP phases include one Dirac phase δ_{CP} and two Majorana phases α_{21} , α_{31} . The former can be measured in neutrino oscillation experiments, the latter are relevant for neutrinoless double beta decay and more difficult to be measured.

neously break the symmetry $S_4 \rtimes H_{CP}$ down to $Z_2 \times H_{CP}^\nu$ in the neutrino sector, which was simply assumed to happen in [20]. In our models, the choice of preserved CP symmetry H_{CP}^ν in the neutrino sector is controlled by free (real) parameters in the flavon potential. We propose two realistic models of this kind, one at the effective level and one at the renormalisable level. Both models predict trimaximal lepton mixing with CP being either fully preserved or maximally broken, with the intermediate possibility forbidden by the structure of the models.

The layout of the rest of this paper is as follows. In section 2 we draw together some of the basic observations about generalised CP symmetry when combined with a discrete family symmetry that are rapidly becoming to be appreciated in the literature. In section 3 we focus on the case of S_4 family symmetry, which was also analysed extensively in [20]. However, in our case, we will work in a different basis, the diagonal charged lepton basis, where the representation matrix for the order three generators T is diagonal, and we verify explicitly that the same physical results emerge, as expected. This discussion also sets the notation and allows us to be rather brief when discussing the physical implications of the models which follow. In section 4 we present our first example of an $S_4 \rtimes H_{CP}$ model at the effective level (i.e. involving non-renormalisable operators) where the flavon potential leads to exactly the kinds of possibilities for CP violation discussed in section 3. In section 5 we present a superior renormalisable $S_4 \rtimes H_{CP}$ model which has the added advantage of explaining why the reactor angle is smaller than the other lepton mixing angles, and also leads to similar options for CP violation. Section 6 concludes the main body of the paper, followed by two appendices on more technical aspects.

2 A consistent definition of generalised CP transformations

In this section, we start by briefly reviewing for completeness how CP transformations are generally defined in a consistent way *without* an existing family symmetry. This is begun by noting that in the extension of the Standard Model (SM) with Majorana masses for the left-handed neutrinos, the mass terms and the charged current interactions for the lepton fields read (below the electroweak symmetry breaking scale),

$$\mathcal{L} = -\bar{l}_L m_l l_R - \frac{1}{2} \nu_L^T C m_\nu \nu_L + \frac{g}{\sqrt{2}} \bar{l}_L \gamma^\mu \nu_L W_\mu^- + h.c. \quad (2.1)$$

$l_{L,R} \equiv (e, \mu, \tau)_{L,R}^T$ stands for the SM charged lepton fields, $\nu_L \equiv (\nu_e, \nu_\mu, \nu_\tau)_L^T$ are the left-handed neutrino fields, C is the charge conjugation matrix, $m_{l,\nu}$ are complex matrices, and m_ν is symmetric. The charged current interactions are invariant under the so-called

generalised CP transformation defined by²

$$\begin{aligned} l_L(x) &\xrightarrow{CP} iX_L\gamma^0 C\bar{l}_L^T(x'), \\ \nu_L(x) &\xrightarrow{CP} iX_L\gamma^0 C\bar{\nu}_L^T(x'), \\ l_R(x) &\xrightarrow{CP} iX_R\gamma^0 C\bar{l}_R^T(x'), \end{aligned} \quad (2.2)$$

where X_L and X_R are unitary matrices acting on generation space and $x' = (t, -\mathbf{x})$. Notice that the “canonical”/traditional CP transformation can be recovered from the above equations by setting both X_L and X_R to the identity matrix. Furthermore, note that the lepton fields l_L and ν_L have to transform in the same way due to the presence of the left-handed charged-current interactions. Then, the Lagrangian of Eq. (2.1) conserves CP if and only if the lepton mass matrices m_l and m_ν satisfy the following relations,

$$X_L^\dagger m_l X_R = m_l^*, \quad X_L^T m_\nu X_L = m_\nu^*. \quad (2.3)$$

Let us now consider a theory that is invariant under both a generalised CP symmetry and a family symmetry G_f , which is the main focus of this paper. We include this discussion in order to set the notation and to make the rest of the paper accessible. However, since all the information in this section is already in the literature, the discussion below is necessarily brief and we refer interested readers to e.g. [17, 20] for more details. Let us, then, consider a field φ in a generic irreducible representation \mathbf{r} of G_f which transforms under the action of G_f as

$$\varphi(x) \xrightarrow{G_f} \rho_{\mathbf{r}}(g)\varphi(x), \quad g \in G_f, \quad (2.4)$$

where $\rho_{\mathbf{r}}(g)$ denotes the representation matrix for the element g in the irreducible representation \mathbf{r} . The mapping of φ under a generalised CP transformation is given by [12–14, 17, 20, 21]:

$$\varphi(x) \xrightarrow{CP} X_{\mathbf{r}}\varphi^*(x'), \quad (2.5)$$

where $X_{\mathbf{r}}$ is a unitary matrix in order to keep the kinetic term invariant, and the obvious action of CP on the spinor indices has been omitted for the case of φ being a spinor.

Requiring a generalised CP symmetry H_{CP} in the context of a flavour model based on some family symmetry G_f , restricts the allowed choices for $X_{\mathbf{r}}$ considerably [17, 20]. If we first perform a CP transformation, followed by a family symmetry transformation, and subsequently an inverse CP transformation we obtain

$$\varphi(x) \xrightarrow{CP} X_{\mathbf{r}}\varphi^*(x') \xrightarrow{G_f} X_{\mathbf{r}}\rho_{\mathbf{r}}^*(g)\varphi^*(x') \xrightarrow{CP^{-1}} X_{\mathbf{r}}\rho_{\mathbf{r}}^*(g)X_{\mathbf{r}}^{-1}\varphi(x). \quad (2.6)$$

As the theory should be invariant under this sequence of transformations, the resulting net transformation must be equivalent to a family symmetry transformation $\rho_{\mathbf{r}}(g')$ of some group element g'

$$X_{\mathbf{r}}\rho_{\mathbf{r}}^*(g)X_{\mathbf{r}}^{-1} = \rho_{\mathbf{r}}(g'), \quad g, g' \in G_f. \quad (2.7)$$

²A factor of i is included, differing from the definition of Ref. [21], in order to eliminate an extraneous “−” sign from the second equation of Eq. (2.3).

This equation defines the so-called *consistency equation* which must be satisfied in order to implement both generalised CP and family symmetries simultaneously. It is important to note that Eq. (2.7) must hold for *all* representations \mathbf{r} simultaneously, i.e. the elements g and g' must be the same for all irreducible representations. Furthermore, Eq. (2.7) implies that the generalised CP transformation $X_{\mathbf{r}}$ maps the group element g onto g' , and this mapping preserves the family symmetry's group structure.³ For faithful representations \mathbf{r} where the function $\rho_{\mathbf{r}}$ maps each element of G_f into a distinct matrix, the consistency equation will define a unique mapping of the abstract group G_f to itself.⁴ It is also clear from Eq. (2.7) that g and g' must be of the same order.

It is important to note that, if $X_{\mathbf{r}}$ is a solution to Eq. (2.7), then not only is $e^{i\theta} X_{\mathbf{r}}$ a solution (with arbitrary phase θ), but also

$$\rho_{\mathbf{r}}(h)X_{\mathbf{r}}, \quad \text{with } h \in G_f, \quad (2.8)$$

are solutions as well.⁵ Therefore the consistency equation can only determine the possible form of the CP transformation $X_{\mathbf{r}}$ up to an overall arbitrary phase and a G_f transformation $\rho_{\mathbf{r}}(h)$ for a given irreducible representation \mathbf{r} . This statement applies to any family symmetry group G_f regardless of it being discrete or continuous. For discrete family symmetries, it is sufficient to impose the consistency equation Eq. (2.7) on the group's generators.

3 $S_4 \rtimes H_{CP}$ scenarios with $Z_2 \times H_{CP}^\nu$ preserved in neutrino sector and Z_3 in the charged lepton sector

In the remainder of the paper, we will investigate the generalised CP transformations H_{CP} consistent with S_4 as a family symmetry group,

$$S_4 \rtimes H_{CP}. \quad (3.1)$$

In other words we shall be interested in theories which (at high energies above the symmetry breaking scale) respects $S_4 \rtimes H_{CP}$ where the fields transform as

$$\varphi(x) \xrightarrow{S_4} \rho_{\mathbf{r}}(g)\varphi(x), \quad g \in S_4, \quad (3.2)$$

$$\varphi(x) \xrightarrow{H_{CP}} X_{\mathbf{r}} \varphi^*(x'), \quad (3.3)$$

³ $X_{\mathbf{r}}\rho_{\mathbf{r}}^*(g_1g_2)X_{\mathbf{r}}^{-1} = X_{\mathbf{r}}\rho_{\mathbf{r}}^*(g_1)\rho_{\mathbf{r}}^*(g_2)X_{\mathbf{r}}^{-1} = X_{\mathbf{r}}\rho_{\mathbf{r}}^*(g_1)X_{\mathbf{r}}^{-1}X_{\mathbf{r}}\rho_{\mathbf{r}}^*(g_2)X_{\mathbf{r}}^{-1} = \rho_{\mathbf{r}}(g'_1)\rho_{\mathbf{r}}(g'_2) = \rho_{\mathbf{r}}(g'_1g'_2)$, where we denote $X_{\mathbf{r}}\rho_{\mathbf{r}}^*(g_i)X_{\mathbf{r}}^{-1} = \rho_{\mathbf{r}}(g'_i)$. Therefore the CP transformation $X_{\mathbf{r}}$ is a homomorphism of the family symmetry group G_f .

⁴The consistency equation defines an automorphism for the group G_f . See Ref. [17] for a more formal treatment.

⁵To see this note that $\rho_{\mathbf{r}}(h)X_{\mathbf{r}}\rho_{\mathbf{r}}^*(g)(\rho_{\mathbf{r}}(h)X_{\mathbf{r}})^{-1} = \rho_{\mathbf{r}}(h)X_{\mathbf{r}}\rho_{\mathbf{r}}^*(g)X_{\mathbf{r}}^{-1}\rho_{\mathbf{r}}^{-1}(h) = \rho_{\mathbf{r}}(h)\rho_{\mathbf{r}}(g')\rho_{\mathbf{r}}(h^{-1}) = \rho_{\mathbf{r}}(hg'h^{-1})$. Therefore the generalised CP transformation $\rho_{\mathbf{r}}(h)X_{\mathbf{r}}$ maps the group element g into $hg'h^{-1}$ which belongs to the conjugacy class of g' . It is equivalent to a CP transformation $X_{\mathbf{r}}$ plus an inner automorphism of the family symmetry group G_f [17].

where the elements of H_{CP} are denoted by $X_{\mathbf{r}}$. Since S_4 is our primary focus, in Appendix B, we prove explicitly the statement in [17] that the most general CP transformation H_{CP} consistent with the S_4 flavour group is simply given by $X_{\mathbf{r}}$ being equal to the identity (up to an inner automorphism). Therefore the most general CP transformation H_{CP} consistent with S_4 family symmetry is of the same form as the family group transformation itself,

$$X_{\mathbf{r}} = \rho_{\mathbf{r}}(h) , \quad h \in S_4 , \quad (3.4)$$

where h can be any of the 24 group elements of S_4 .

Following [20], we further assume in this section that the underlying combined symmetry group $S_4 \rtimes H_{CP}$ is broken to G_{CP}^ν in the neutrino sector and G^ℓ in the charged lepton sector⁶

$$G_{CP}^\nu \cong Z_2 \times H_{CP}^\nu \quad \text{and} \quad G^\ell \cong Z_3 . \quad (3.5)$$

The main purpose of this paper (discussed in the following sections) is to show how the original symmetry $S_4 \rtimes H_{CP}$ can be spontaneously broken to $Z_2 \times H_{CP}^\nu$ in the neutrino sector, which goes beyond the analysis in [20]. However, in this section, we begin by following in the footsteps of [20] but using a different S_4 basis, namely that in which the charged leptons are diagonal and all the mixing arises from the neutrino sector. In fact we regard it as necessary to repeat the symmetry analysis in the new basis in order to verify explicitly that the two bases give equivalent results, thereby placing our later results on a sound footing. It is also useful to set the notation and to allow us to arrive at the physical results very efficiently in the next sections. Therefore, although the results in this section may seem like repetition, we regard it as both necessary and useful to proceed by first following the analysis in [20] in a different basis.

The basis in which we work in this paper is motivated by the trimaximal S_4 model of [22] (originally proposed without H_{CP}), where the resulting symmetry breaking led to a preserved Z_2^S in the neutrino sector generated by S , as well as a diagonal charged lepton mass matrix which arises thanks to the chosen S_4 basis with diagonal complex T generator, as discussed in Appendix B. The Z_2^S symmetry of the neutrino sector will automatically produce a neutrino mass matrix in which the second column of the corresponding mixing matrix is proportional to $(1, 1, 1)^T$. In the next section we shall discuss models which are inspired by this model but include H_{CP} and are hence based on $S_4 \rtimes H_{CP}$ which is spontaneously broken to

$$G_{CP}^\nu \cong Z_2^S \times H_{CP}^\nu , \quad (3.6)$$

in the neutrino sector. For G_{CP}^ν to be a well-defined symmetry, the consistency relation of Eq. (2.7) needs to be satisfied for the residual symmetry group Z_2^S . In other words the elements of H_{CP}^ν must satisfy,

$$X_{\mathbf{r}} \rho_{\mathbf{r}}^*(S) X_{\mathbf{r}}^{-1} = \rho_{\mathbf{r}}(S). \quad (3.7)$$

One can arrive at the above restricted consistency equation by recalling the general consistency condition of Eq. (2.7), with $g, g' \in Z_2^S$, and realising that g' must be S , as it is

⁶In the models constructed in sections 4 and 5, the Z_3 subgroup of S_4 will be broken. However, this happens in such a way that the resulting charged lepton mass matrix remains diagonal.

the only element which (trivially) has the same order as $g = S$. Armed with this information, recall from section 2 that a faithful representation of the group G_f must be used to uniquely determine the mapping of the consistency equation (then the unfaithful representations will follow). Hence, we proceed by considering faithful representations of S_4 , i.e. the three dimensional irreducible representations.

For the triplets, one can easily show that for H_{CP}^ν there are only eight choices for the $X_{\mathbf{r}}$ in Eq. (3.7) which are acceptable:

$$X_{\mathbf{r}} = \rho_{\mathbf{r}}(h) , \quad (3.8)$$

with

$$h \in \{1, S, TST^2, T^2ST, U, SU, TST^2U, T^2STU\} . \quad (3.9)$$

This may be compared to the 24 choices for $X_{\mathbf{r}}$ in Eq. (3.4) corresponding to the original H_{CP} before it is broken (in the charged lepton sector). Note that the condition in Eq. (3.7) is automatically satisfied for the doublet and the singlet representations (the unfaithful representations), as is implied by the consistency equation. Comparing the results of Eq. (3.9) to the six solutions of the analysis given in [20], we observe that our third and fourth solutions of Eq. (3.9) are missing in [20]. This is because the authors of [20] have made the assumption that a generalised CP transformation must be symmetric. However, it is not necessary to demand symmetric $X_{\mathbf{r}}$ if there is a family symmetry, so we keep all eight solutions in the following.

Let us assume that the three generations of lepton doublets reside in the **3**-dimensional irreducible representation.⁷ Then, it is possible to construct the light neutrino mass matrix m_ν by demanding that it respects both the residual flavour symmetry Z_2^S and the generalised CP symmetry H_{CP}^ν :

$$\rho_{\mathbf{3}}^T(S)m_\nu\rho_{\mathbf{3}}(S) = m_\nu, \quad (3.10)$$

$$X_{\mathbf{3}}^T m_\nu X_{\mathbf{3}} = m_\nu^*. \quad (3.11)$$

The most general matrix which satisfies Eq. (3.10) has the form

$$m_\nu = \alpha \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} + \beta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad (3.12)$$

where α, β, γ , and ϵ are complex parameters. The CP symmetry of Eq. (3.11) will further constrain them to be either real or imaginary. The eight possibilities listed in Eq. (3.9) are classified in Table 1 according to their four distinct predictions for α, β, γ , and ϵ being real or imaginary.

In order to diagonalise m_ν of Eq. (3.12), we perform a tri-bimaximal transformation U_{TB} ,

$$m'_\nu = U_{TB}^T m_\nu U_{TB} = \begin{pmatrix} 3\alpha + \beta - \gamma & 0 & -\sqrt{3}\epsilon \\ 0 & \beta + 2\gamma & 0 \\ -\sqrt{3}\epsilon & 0 & 3\alpha - \beta + \gamma \end{pmatrix}, \quad (3.13)$$

⁷The results would be the same if the lepton doublets were assigned to $\mathbf{3}'$, since the representation $\mathbf{3}'$ differs from $\mathbf{3}$ only in the overall sign of the generator U .

	H_{CP}^ν	α	β	γ	ϵ
(i)	$\rho_{\mathbf{r}}(1), \rho_{\mathbf{r}}(S)$	real	real	real	real
(ii)	$\rho_{\mathbf{r}}(TST^2), \rho_{\mathbf{r}}(T^2ST)$	imaginary	real	real	real
(iii)	$\rho_{\mathbf{r}}(U), \rho_{\mathbf{r}}(SU)$	real	real	real	imaginary
(iv)	$\rho_{\mathbf{r}}(TST^2U), \rho_{\mathbf{r}}(T^2STU)$	imaginary	real	real	imaginary

Table 1: The generalised CP transformations consistent with a residual Z_2^S symmetry in the neutrino sector and their corresponding constraints on α , β , γ and ϵ parameters in Eq. (3.12).

with

$$U_{TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad (3.14)$$

followed by a (1,3) rotation U'_ν ,

$$U_\nu'^T m'_\nu U'_\nu = \text{diag}(m_1, m_2, m_3). \quad (3.15)$$

Then the PMNS matrix is

$$U_{PMNS} = U_{TB} U'_\nu, \quad (3.16)$$

where U_{PMNS} is only determined up to the permutations of row and column, since the order of both the charged lepton and neutrino masses cannot be predicted in the present symmetry guided approach, see also [20]. Finally, we shall work in the PDG convention [7],

$$U_{PMNS} = V \text{diag}(1, e^{i\frac{\alpha_{21}}{2}}, e^{i\frac{\alpha_{31}}{2}}), \quad (3.17)$$

with

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{CP}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{CP}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{CP}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{CP}} & c_{23}c_{13} \end{pmatrix}. \quad (3.18)$$

We are now prepared to consider each of the four cases of Table 1 in turn.

(i) $H_{CP}^\nu = \rho_{\mathbf{r}}(1), \rho_{\mathbf{r}}(S)$

In this case, the unitary matrix U'_ν is

$$U'_\nu = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} P, \quad (3.19)$$

with

$$\tan 2\theta = \frac{\sqrt{3} \epsilon}{\beta - \gamma}. \quad (3.20)$$

The transformation P in Eq. (3.19) is a unitary diagonal matrix with entries ± 1 or $\pm i$, which renders the light neutrino masses $m_{1,2,3}$ positive. In the following cases, we shall omit this trivial factor. Given the diagonal charged lepton mass matrix, the leptonic mixing matrix is

$$U_{PMNS} = U_{TB} U'_\nu = \begin{pmatrix} \sqrt{\frac{2}{3}} \cos \theta & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \sin \theta \\ -\frac{1}{\sqrt{6}} \cos \theta - \frac{1}{\sqrt{2}} \sin \theta & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \sin \theta + \frac{1}{\sqrt{2}} \cos \theta \\ -\frac{1}{\sqrt{6}} \cos \theta + \frac{1}{\sqrt{2}} \sin \theta & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \sin \theta - \frac{1}{\sqrt{2}} \cos \theta \end{pmatrix} P. \quad (3.21)$$

As a result, the lepton mixing angles and CP phases take the values

$$\sin \alpha_{21} = \sin \alpha_{31} = \sin \delta_{CP} = 0, \quad (3.22)$$

$$\sin^2 \theta_{13} = \frac{2}{3} \sin^2 \theta, \quad \sin^2 \theta_{12} = \frac{1}{2 + \cos 2\theta}, \quad \sin^2 \theta_{23} = \frac{1}{2} \left[1 + \frac{\sqrt{3} \sin 2\theta}{2 + \cos 2\theta} \right]. \quad (3.23)$$

Notice that there is no CP violation in this case, but it is otherwise a viable scenario. Finally, the light neutrino masses are

$$\begin{aligned} m_1 &= \left| 3\alpha - \sqrt{(\gamma - \beta)^2 + 3\epsilon^2} \right|, \\ m_2 &= |\beta + 2\gamma|, \\ m_3 &= \left| 3\alpha + \sqrt{(\gamma - \beta)^2 + 3\epsilon^2} \right|. \end{aligned} \quad (3.24)$$

(ii) $H_{CP}^\nu = \rho_{\mathbf{r}}(TST^2), \rho_{\mathbf{r}}(T^2ST)$

This case is not discussed in [20] because the authors require that the CP transformation be both unitary and symmetric. However, it turns out that for this case

$$m'_\nu m'^\dagger_\nu = \begin{pmatrix} -9\alpha^2 + (\beta - \gamma)^2 + 3\epsilon^2 & 0 & 0 \\ 0 & (\beta + 2\gamma)^2 & 0 \\ 0 & 0 & -9\alpha^2 + (\beta - \gamma)^2 + 3\epsilon^2 \end{pmatrix},$$

which implies that $|m_1|^2 = |m_3|^2$. Therefore, the resulting light neutrino mass spectrum is (partially) degenerate, and the PMNS matrix cannot be determined uniquely. Phenomenologically, this case is clearly not viable.

(iii) $H_{CP}^\nu = \rho_{\mathbf{r}}(U), \rho_{\mathbf{r}}(SU)$

The unitary transformation U'_ν is of the form

$$U'_\nu = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -i \sin \theta & 0 & i \cos \theta \end{pmatrix}, \quad (3.25)$$

with the angle θ

$$\tan 2\theta = \frac{i\epsilon}{\sqrt{3}\alpha}. \quad (3.26)$$

The PMNS matrix takes the form

$$U_{PMNS} = U_{TB}U'_\nu = \begin{pmatrix} \sqrt{\frac{2}{3}}\cos\theta & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}}\sin\theta \\ -\frac{1}{\sqrt{6}}\cos\theta + \frac{i}{\sqrt{2}}\sin\theta & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}}\sin\theta - \frac{i}{\sqrt{2}}\cos\theta \\ -\frac{1}{\sqrt{6}}\cos\theta - \frac{i}{\sqrt{2}}\sin\theta & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}}\sin\theta + \frac{i}{\sqrt{2}}\cos\theta \end{pmatrix}. \quad (3.27)$$

Therefore the lepton mixing parameters are

$$\sin\alpha_{21} = 0, \quad \sin\alpha_{31} = 0, \quad |\sin\delta_{CP}| = 1, \quad (3.28)$$

$$\sin^2\theta_{13} = \frac{2}{3}\sin^2\theta, \quad \sin^2\theta_{12} = \frac{1}{2+\cos 2\theta}, \quad \sin^2\theta_{23} = \frac{1}{2}. \quad (3.29)$$

Notice that we have maximal Dirac CP violation $\delta_{CP} = \pm\frac{\pi}{2}$. However, the sign cannot be fixed uniquely because it depends on the value of the angle θ and the order of the light neutrino masses. This case provides a maximal atmospheric mixing angle, with the light neutrino masses given by

$$\begin{aligned} m_1 &= \left| \beta - \gamma - \sqrt{9\alpha^2 - 3\epsilon^2} \right|, \\ m_2 &= |\beta + 2\gamma|, \\ m_3 &= \left| \beta - \gamma + \sqrt{9\alpha^2 - 3\epsilon^2} \right|. \end{aligned} \quad (3.30)$$

Therefore this case is viable as well.

(iv) $H_{CP}^\nu = \rho_r(TST^2U)$, $\rho_r(T^2STU)$

In this case, the unitary matrix U'_ν is given by

$$U'_\nu = \frac{1}{\sqrt{2}} \begin{pmatrix} -e^{i(\theta+\frac{\pi}{4})} & 0 & -e^{i(\theta-\frac{\pi}{4})} \\ 0 & 1 & 0 \\ e^{-i(\theta-\frac{\pi}{4})} & 0 & -e^{-i(\theta+\frac{\pi}{4})} \end{pmatrix}, \quad (3.31)$$

with

$$\tan 2\theta = \frac{-\beta + \gamma}{3i\alpha}. \quad (3.32)$$

The resulting PMNS matrix reads

$$\overline{U}_{PMNS} = U_{TB}U'_\nu = \begin{pmatrix} -\frac{1}{\sqrt{3}}e^{i(\theta+\frac{\pi}{4})} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}}e^{i(\theta-\frac{\pi}{4})} \\ \frac{1}{2\sqrt{3}}e^{i(\theta+\frac{\pi}{4})} - \frac{1}{2}e^{-i(\theta-\frac{\pi}{4})} & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}}e^{i(\theta-\frac{\pi}{4})} + \frac{1}{2}e^{-i(\theta+\frac{\pi}{4})} \\ \frac{1}{2\sqrt{3}}e^{i(\theta+\frac{\pi}{4})} + \frac{1}{2}e^{-i(\theta-\frac{\pi}{4})} & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}}e^{i(\theta-\frac{\pi}{4})} - \frac{1}{2}e^{-i(\theta+\frac{\pi}{4})} \end{pmatrix}. \quad (3.33)$$

Agreement with the present experimental data can be achieved if we permute the rows of the above \overline{U}_{PMNS} by [20]

$$U_{PMNS}^{1st} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \overline{U}_{PMNS}, \quad \text{or} \quad U_{PMNS}^{2nd} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \overline{U}_{PMNS}, \quad (3.34)$$

to yield two phenomenologically viable PMNS matrices, U_{PMNS}^{1st} and U_{PMNS}^{2nd} , so named because they differ only in their prediction of the atmospheric mixing angle. U_{PMNS}^{1st} predicts the atmospheric mixing angle to be in the first octant and U_{PMNS}^{2nd} in the second. The three lepton mixing angles are then determined to be ⁸

$$\begin{aligned} \sin^2 \theta_{13} &= \frac{1}{6} \left(2 - \sqrt{3} \cos 2\theta \right), & \sin^2 \theta_{12} &= \frac{2}{4 + \sqrt{3} \cos 2\theta}, \\ \sin^2 \theta_{23}^{1st} &= \frac{2}{4 + \sqrt{3} \cos 2\theta}, & \text{or} & \sin^2 \theta_{23}^{2nd} = 1 - \frac{2}{4 + \sqrt{3} \cos 2\theta}. \end{aligned} \quad (3.35)$$

Both solutions share the phase predictions,

$$\begin{aligned} |\sin \alpha_{21}| &= \left| \frac{\sqrt{3} + 2 \cos 2\theta}{2 + \sqrt{3} \cos 2\theta} \right|, & |\sin \alpha'_{31}| &= \left| \frac{4\sqrt{3} \sin 2\theta}{5 - 3 \cos 4\theta} \right|, \\ |\sin \delta_{CP}| &= \left| \frac{\sqrt{4 - 2\sqrt{3} \cos 2\theta} (4 + \sqrt{3} \cos 2\theta) \sin 2\theta}{5 - 3 \cos 4\theta} \right|. \end{aligned} \quad (3.36)$$

where $\alpha'_{31} = \alpha_{31} - 2\delta_{CP}$, the parameter α'_{31} has been redefined to include δ_{CP} which is useful in the context of neutrinoless double beta decay [21]. Lastly, we find the light neutrino masses for this case to be

$$\begin{aligned} m_1 &= \left| \sqrt{3} i\epsilon - \sqrt{(\beta - \gamma)^2 - 9\alpha^2} \right|, \\ m_2 &= |\beta + 2\gamma|, \\ m_3 &= \left| \sqrt{3} i\epsilon + \sqrt{(\beta - \gamma)^2 - 9\alpha^2} \right|. \end{aligned} \quad (3.37)$$

Therefore this case is also viable and involves non-trivial CP violating phases.

4 An effective $S_4 \rtimes H_{CP}$ model

Field	L	N^c	e^c	μ^c	τ^c	$H_{u,d}$	φ_T	η	φ_S	ξ	ϕ	φ_T^0	ζ^0	φ_S^0	ξ^0
S_4	3	3	1	1'	1	1	3	2	3'	1	2	3'	1	3'	1
Z_4	1	1	i	-1	$-i$	1	i	i	1	1	1	-1	-1	1	1
Z_3	ω^2	ω	ω	ω	ω	1	1	1	ω	ω	ω	1	1	ω	ω
$U(1)_R$	1	1	1	1	1	0	0	0	0	0	0	2	2	2	2

Table 2: The particle content and the transformation properties under the family symmetry $S_4 \times Z_4 \times Z_3$ and $U(1)_R$.

In this section, we present a realistic effective model of leptons based on

$$S_4 \rtimes H_{CP}, \quad (4.1)$$

⁸The following results for the mixing angles and CP phases match those of [20] after making the replacement $\theta \rightarrow \pi/4 - \theta$.

supplemented by the extra symmetries

$$Z_4 \times Z_3 \times U(1)_R \quad (4.2)$$

in order to control the allowed operators in the model.

The three generations of left-handed (LH) lepton doublets L and the three generations of right-handed (RH) neutrinos N^c are both unified into the S_4 triplet representation $\mathbf{3}$, while the RH charged leptons are assigned to be the singlet representations $\mathbf{1}$ or $\mathbf{1}'$. The complete list of lepton, Higgs, flavon and driving fields as well as their transformation properties under the family symmetry are listed in Table 2. Notice that the Z_3 symmetry is used to separate the flavons entering the charged lepton sector at leading order (LO) from those of the LO neutrino sector. They are further distinguished by an auxiliary Z_4 symmetry. This Z_4 symmetry is also helpful in achieving the charged lepton mass hierarchies. As discussed in the previous section, the S_4 family symmetry will be spontaneously broken to generate trimaximal mixing.

Furthermore, we will impose a generalised CP symmetry H_{CP} consistent with S_4 as discussed in the previous section. In addition to forcing the coupling constants in the superpotential to be real, the generalised CP symmetry will also be spontaneously broken in the charged lepton sector. In the neutrino sector, a restricted generalised CP symmetry remains,

$$G_{CP}^\nu \cong Z_2^S \times H_{CP}^\nu, \quad (4.3)$$

where this residual symmetry results in predictions for δ_{CP} and in other relations as discussed in section 3. However, unlike in the previous section, the dynamics of the spontaneous breaking of $S_4 \rtimes H_{CP}$ will be discussed by studying the vacuum alignment of a suitable flavon potential to which we now turn.

4.1 Vacuum alignment

The general driving superpotential invariant under the symmetry of the model is

$$\begin{aligned} w_d = & g_1 (\varphi_T^0 (\varphi_T \varphi_T)_{\mathbf{3}'})_{\mathbf{1}} + g_2 (\varphi_T^0 (\eta \varphi_T)_{\mathbf{3}'})_{\mathbf{1}} + g_3 \zeta^0 (\varphi_T \varphi_T)_{\mathbf{1}} + g_4 \zeta^0 (\eta \eta)_{\mathbf{1}} \\ & + f_1 (\varphi_S^0 (\varphi_S \varphi_S)_{\mathbf{3}'})_{\mathbf{1}} + f_2 (\varphi_S^0 (\phi \varphi_S)_{\mathbf{3}'})_{\mathbf{1}} + f_3 (\varphi_S^0 \varphi_S)_{\mathbf{1}} \xi \\ & + h_1 \xi^0 (\varphi_S \varphi_S)_{\mathbf{1}} + h_2 \xi^0 (\phi \phi)_{\mathbf{1}} + h_3 \xi^0 \xi^2, \end{aligned} \quad (4.4)$$

where $(\dots)_{\mathbf{r}}$ denotes the contraction of the S_4 indices to the representation \mathbf{r} . If we require the theory to be invariant under the generalised CP transformation defined above, then all the couplings g_i , f_i and h_i are real parameters. In the SUSY limit, the vacuum configuration is determined by the vanishing of the derivative of the driving superpotential w_d with respect to each component of the driving fields. The vacuum in the charged lepton

sector is determined by

$$\begin{aligned}
\frac{\partial w_d}{\partial \varphi_{T_1}^0} &= 2g_1(\varphi_{T_1}^2 - \varphi_{T_2}\varphi_{T_3}) + g_2(\eta_1\varphi_{T_2} - \eta_2\varphi_{T_3}) = 0 , \\
\frac{\partial w_d}{\partial \varphi_{T_2}^0} &= 2g_1(\varphi_{T_2}^2 - \varphi_{T_1}\varphi_{T_3}) + g_2(\eta_1\varphi_{T_1} - \eta_2\varphi_{T_2}) = 0 , \\
\frac{\partial w_d}{\partial \varphi_{T_3}^0} &= 2g_1(\varphi_{T_3}^2 - \varphi_{T_1}\varphi_{T_2}) + g_2(\eta_1\varphi_{T_3} - \eta_2\varphi_{T_1}) = 0 , \\
\frac{\partial w_d}{\partial \zeta^0} &= g_3(\varphi_{T_1}^2 + 2\varphi_{T_2}\varphi_{T_3}) + 2g_4\eta_1\eta_2 = 0 .
\end{aligned} \tag{4.5}$$

This set of equations admits two solutions, the first one is given by

$$\langle \varphi_T \rangle = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} v_T, \quad \langle \eta \rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix} v_\eta, \quad \text{with} \quad v_T^2 = -\frac{2g_4}{3g_3}v_\eta^2, \tag{4.6}$$

and the second solution reads

$$\langle \varphi_T \rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} v_T, \quad \langle \eta \rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} v_\eta, \quad \text{with} \quad v_T = \frac{g_2}{2g_1}v_\eta. \tag{4.7}$$

Here, we choose the second solution.⁹ The phase of v_η can be absorbed into the lepton fields, therefore we can take v_η to be real, then the VEV v_T is real as well. The equations determining the vacuum alignment in the neutrino sector are

$$\begin{aligned}
\frac{\partial w_d}{\partial \varphi_{S_1}^0} &= 2f_1(\varphi_{S_1}^2 - \varphi_{S_2}\varphi_{S_3}) + f_2(\phi_1\varphi_{S_2} + \phi_2\varphi_{S_3}) + f_3\xi\varphi_{S_1} = 0 , \\
\frac{\partial w_d}{\partial \varphi_{S_2}^0} &= 2f_1(\varphi_{S_2}^2 - \varphi_{S_1}\varphi_{S_3}) + f_2(\phi_1\varphi_{S_1} + \phi_2\varphi_{S_2}) + f_3\xi\varphi_{S_3} = 0 , \\
\frac{\partial w_d}{\partial \varphi_{S_3}^0} &= 2f_1(\varphi_{S_3}^2 - \varphi_{S_1}\varphi_{S_2}) + f_2(\phi_1\varphi_{S_3} + \phi_2\varphi_{S_1}) + f_3\xi\varphi_{S_2} = 0 , \\
\frac{\partial w_d}{\partial \xi^0} &= h_1(\varphi_{S_1}^2 + 2\varphi_{S_2}\varphi_{S_3}) + 2h_2\phi_1\phi_2 + h_3\xi^2 = 0 .
\end{aligned} \tag{4.8}$$

Disregarding the ambiguity caused by the S_4 family symmetry transformations, we find the solution

$$\langle \varphi_S \rangle = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} v_S, \quad \langle \phi \rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad \langle \xi \rangle = u, \tag{4.9}$$

where the VEVs obey the relation

$$v_1 + v_2 = -f_3u/f_2 \equiv -Bu, \tag{4.10}$$

$$3h_1v_S^2 + 2h_2v_1v_2 = -h_3u^2. \tag{4.11}$$

⁹Notice that the first solution can be eliminated by adding another driving field with the same flavour numbers as ζ^0 [23].

The VEVs v_S , v_1 and v_2 are undetermined. In order to solve this problem, we introduce a second driving fields with identical quantum numbers as ξ^0 [23]. Consequently we obtain two F -term conditions which are identical in their structures but involve independent coupling constants, h_i for one driving field and h'_i for the other, i.e.

$$3h'_1 v_S^2 + 2h'_2 v_1 v_2 = -h'_3 u^2 . \quad (4.12)$$

As both conditions, Eq. (4.11) and Eq. (4.12), must be satisfied, one can find a unique solution for the VEV of the φ_S flavon

$$v_S^2 = \frac{h_2 h'_3 - h_3 h'_2}{3(h_1 h'_2 - h_2 h'_1)} u^2 , \quad (4.13)$$

and

$$v_1 v_2 = \frac{h_3 h'_1 - h_1 h'_3}{2(h_1 h'_2 - h_2 h'_1)} u^2 \equiv C u^2 . \quad (4.14)$$

Since it is always possible to absorb the phase of u by a redefinition of the matter fields, we can take u to be real without loss of generality, then v_S is either real or purely imaginary. The vacuum of the doublet ϕ can be obtained by solving Eq. (4.10) together with Eq. (4.14).

- For $B^2 - 4C > 0$ we find

$$\begin{cases} v_1 = \frac{1}{2} \left[-B \pm \sqrt{B^2 - 4C} \right] u , \\ v_2 = \frac{1}{2} \left[-B \mp \sqrt{B^2 - 4C} \right] u , \end{cases} \quad (4.15)$$

from which we see that v_1 and v_2 are both real.

- For $B^2 - 4C < 0$ we obtain

$$\begin{cases} v_1 = \frac{1}{2} \left[-B \pm i\sqrt{4C - B^2} \right] u , \\ v_2 = \frac{1}{2} \left[-B \mp i\sqrt{4C - B^2} \right] u , \end{cases} \quad (4.16)$$

which entails $v_1 = v_2^*$ in this case.

4.2 The lepton masses and mixings

The effective superpotential for the charged lepton masses which is allowed by the symmetries is given as

$$\begin{aligned} w_\ell = & \frac{y_\tau}{\Lambda} (L\varphi_T)_1 H_d \tau^c + \frac{y_{\mu_1}}{\Lambda^2} (L(\varphi_T \varphi_T)_{\mathbf{3}'})_1 H_d \mu^c + \frac{y_{\mu_2}}{\Lambda^2} (L(\eta \varphi_T)_{\mathbf{3}'})_1 H_d \mu^c \\ & + \sum_i \frac{y_{e_i}}{\Lambda^3} (L\mathcal{O}_i)_1 H_d e^c , \end{aligned} \quad (4.17)$$

where \mathcal{O} stands for

$$\mathcal{O} = \{ \varphi_T^3, \eta \varphi_T^2, \eta^2 \varphi_T \} , \quad (4.18)$$

and all possible S_4 contractions are to be considered. It is interesting to note that the tau mass is suppressed by $1/\Lambda$, while the muon and electron masses appear at order $1/\Lambda^2$ and $1/\Lambda^3$, respectively. The mass hierarchies among the charged lepton are thus produced in a natural way. Inserting the vacuum alignment of Eq. (4.7), we find a diagonal charged lepton mass matrix with

$$m_\tau = y_\tau \frac{v_T}{\Lambda} v_d, \quad m_\mu = \left[2y_{\mu_1} \left(\frac{v_T}{\Lambda} \right)^2 - y_{\mu_2} \frac{v_\eta v_T}{\Lambda^2} \right] v_d, \quad m_e = y_e \left(\frac{v_T}{\Lambda} \right)^3 v_d, \quad (4.19)$$

where y_e is the result five different contributions corresponding to the y_{e_i} in Eq. (4.17). Obviously the VEVs of the flavons φ_T and η are responsible for the spontaneous breaking of both the family symmetry and the generalised CP symmetry. It is straightforward to check that S_4 is broken completely in the charged lepton sector. In addition, only one CP symmetry $X_{\mathbf{r}} = \rho_{\mathbf{r}}(1)$ out of the 24 consistent CP transformations is preserved by the flavons φ_T and η , given the previously mentioned fact that both v_T and v_η can be chosen to be real. As a consequence, the residual CP symmetry in the charged lepton sector is $H_{CP}^\ell = \rho_{\mathbf{r}}(1)$. Now we turn to the neutrino sector, the effective superpotential is of the form

$$w_\nu = y(LN^c)_{\mathbf{1}} H_u + y_1((N^c N^c)_{\mathbf{3}'}) \varphi_S + y_2(N^c N^c)_{\mathbf{1}} \xi + y_3((N^c N^c)_{\mathbf{2}}) \phi)_{\mathbf{1}}, \quad (4.20)$$

where all the couplings y_i are real due to the general CP invariance. The Dirac neutrino mass matrix takes a trivial form

$$m_D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} y v_u. \quad (4.21)$$

Given the vacuum configuration of Eq. (4.9), it is straightforward to derive the RH neutrino mass matrix

$$m_M = a \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} + b \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (4.22)$$

where we have introduced the parameters

$$a = y_1 v_S, \quad b = y_2 u, \quad c = y_3 v_2, \quad d = y_3 v_1. \quad (4.23)$$

The light neutrino mass matrix is then obtained from the seesaw formula, yielding

$$\begin{aligned} m_\nu &= -m_D m_M^{-1} m_D^T \\ &= \alpha \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} + \beta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}. \end{aligned} \quad (4.24)$$

This matrix is of the same form as the neutrino mass matrix in Eq. (3.12) which is the most general matrix invariant under the action of the S_4 element S . The parameters α ,

		α	β	γ	ϵ
(i)	$a \in \mathbb{R}, b \in \mathbb{R}, c \in \mathbb{R}, d \in \mathbb{R}$	real	real	real	real
(ii)	$a \in \mathbb{C}, b \in \mathbb{R}, c \in \mathbb{R}, d \in \mathbb{R}$ with $a = -a^*$	imaginary	real	real	real
(iii)	$a \in \mathbb{R}, b \in \mathbb{R}, c \in \mathbb{C}, d \in \mathbb{C}$ with $c = d^*$	real	real	real	imaginary
(iv)	$a \in \mathbb{C}, b \in \mathbb{R}, c \in \mathbb{C}, d \in \mathbb{C}$ with $a = -a^*$ and $c = d^*$	imaginary	real	real	imaginary

Table 3: The four possible cases of the effective model which are controlled by the input parameters of the flavon potential.

β , γ and ϵ are related to a , b , c and d by

$$\begin{aligned}
\alpha &= \frac{a}{-9a^2 + b^2 + (c+d)^2 - 3cd - b(c+d)} , \\
\beta &= \frac{3a^2 - b^2 + cd}{(b+c+d)[-9a^2 + b^2 + (c+d)^2 - 3cd - b(c+d)]} , \\
\gamma &= \frac{6a^2 - (c+d)^2 + 2cd + b(c+d)}{2(b+c+d)[-9a^2 + b^2 + (c+d)^2 - 3cd - b(c+d)]} , \\
\epsilon &= \frac{d-c}{2[-9a^2 + b^2 + (c+d)^2 - 3cd - b(c+d)]} ,
\end{aligned} \tag{4.25}$$

where the overall factor $y^2 v_u^2$ in these expressions has been omitted. From the discussion of the vacuum alignment, we know that b is a real parameter, and a can be real or purely imaginary. For the doublet flavon ϕ , the VEVs v_1 and v_2 can be both real for the first solution in Eq. (4.15), yielding real c and d , or, for the second solution shown in Eq. (4.16), v_1 and v_2 are complex conjugates of each other, yielding $c = d^*$. Therefore there are only four possible cases allowed in our model, which are listed in Table 3.

In order to understand the phenomenological implications of the model, it is useful to know how the generalised CP symmetry is spontaneously broken and what the remnant CP symmetry is. Once the flavon fields acquire their VEVs, the S_4 family symmetry and the generalised CP symmetry are spontaneously broken. Imagining that the flavon VEVs could transform in the way as the flavon fields under the action of CP transformation, i.e.

$$\begin{aligned}
\langle \varphi_S \rangle &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} v_S \xrightarrow{CP} \rho_{\mathbf{3}'}(g) \langle \varphi_S \rangle^* = \rho_{\mathbf{3}'}(g) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} v_S^* , \quad g \in S_4 , \\
\langle \phi \rangle &= \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \xrightarrow{CP} \rho_{\mathbf{2}}(g) \langle \phi \rangle^* = \rho_{\mathbf{2}}(g) \begin{pmatrix} v_1^* \\ v_2^* \end{pmatrix} , \quad \langle \xi \rangle = u \xrightarrow{CP} \langle \xi \rangle^* = u^* ,
\end{aligned} \tag{4.26}$$

then all the initial 24 consistent CP symmetries would be kept. However, the flavon VEVs are only numbers and they don't change at all under a CP transformation. Therefore only those CP transformations which transform the corresponding flavon VEVs into themselves remain symmetries of the theory after symmetry breaking. That means, the residual CP symmetry $\rho_{\mathbf{r}}(g)$ in the neutrino sector satisfies

$$\rho_{\mathbf{3}'}(g) \langle \varphi_S \rangle^* = \langle \varphi_S \rangle, \quad \rho_{\mathbf{2}}(g) \langle \phi \rangle^* = \langle \phi \rangle, \quad \rho_{\mathbf{1}}(g) \langle \xi \rangle^* = \langle \xi \rangle^* = \langle \xi \rangle . \tag{4.27}$$

In the following, we discuss the four cases in Table 3 one by one.

- (i) In this case, both the triplet VEV v_S and the doublet VEVs v_1 and v_2 are real. We can easily check that the generalised CP symmetries for $h = 1$ and $h = S$ are preserved in the neutrino sector, i.e. the residual CP symmetry in the neutrino sector is $H_{CP}^\nu = \{\rho_{\mathbf{r}}(1), \rho_{\mathbf{r}}(S)\}$. From Eq. (4.25) we see that all four parameters α, β, γ and ϵ are real. This is exactly the case (i) discussed in the general analysis of section 3 which is solely based on symmetry arguments. As a result, the PMNS matrix and the resulting lepton mixing parameters are of the form shown in Eqs. (3.21-3.23). In the present case, the generalised CP symmetry $X_{\mathbf{r}} = \rho_{\mathbf{r}}(1)$ is preserved in both the neutrino and the charged lepton sector. According to the general results for weak basis invariants stated in Appendix A, the CP phases would be trivial, i.e. although this case is viable, there is no CP violation, as has been shown already in Eq. (3.22).
- (ii) In this case, v_S is purely imaginary and the remaining VEVs v_1 and v_2 are real. The generalised CP symmetry is broken to $H_{CP}^\nu = \{\rho_{\mathbf{r}}(TST^2), \rho_{\mathbf{r}}(T^2ST)\}$ in the neutrino sector. Concerning the light neutrino mass matrix, the parameter α is imaginary, while β, γ and ϵ are all real. This is exactly case (ii) of the general analysis of section 3, where the light neutrino masses are degenerate and hence this case is not viable.
- (iii) This case corresponds to the VEV v_S being real together with the solution $v_1 = v_2^*$ for the doublet ϕ . Now only two of the 24 generalised CP symmetries are preserved in the neutrino sector and $H_{CP}^\nu = \{\rho_{\mathbf{r}}(U), \rho_{\mathbf{r}}(SU)\}$. The light neutrino mass matrix is of the form given in Eq. (4.24), where α, β and γ are real while ϵ is purely imaginary. This corresponds to case (iii) studied in section 3, and thus the predictions for the PMNS matrix in Eq. (3.27) with lepton mixing parameters as in Eqs. (3.28,3.29). The neutrino mixing is of trimaximal form, and it is remarkable that we obtain maximal Dirac CP violation $\delta_{CP} = \pm\pi/2$ in this case. As for the generalised CP symmetry breaking, although the CP symmetries $H_{CP}^\ell = \{\rho_{\mathbf{r}}(1)\}$ and $H_{CP}^\nu = \{\rho_{\mathbf{r}}(U), \rho_{\mathbf{r}}(SU)\}$ are preserved in the charged lepton and the neutrino sector, respectively, the CP symmetry is completely broken in the full theory, and the mismatch between H_{CP}^ℓ and H_{CP}^ν is precisely the origin of the maximal Dirac CP violation. Therefore this case is viable and predicts maximal CP violation.
- (iv) This case corresponds to v_S being purely imaginary combined with the solution $v_1 = v_2^*$. The remnant CP symmetry in the neutrino sector becomes $H_{CP}^\nu = \{\rho_{\mathbf{r}}(TST^2U), \rho_{\mathbf{r}}(T^2STU)\}$. The parameters α and ϵ of the corresponding light neutrino mass matrix are imaginary, while β and γ are real. This is exactly the case (iv) investigated in the general analysis of section 3. As has been pointed out, in order to achieve agreement with the present experimental data, one has to permute the rows of U_{PMNS} as done in Eq. (3.34). This permutation corresponds to exchanging the three charged lepton masses. However, in the present model, the charged lepton masses are predicted to be of different orders in the expansion parameter v_T/Λ (v_η/Λ), so that this permutation is forbidden. Although we can permute its columns, as the neutrino mass spectrum can be either normal or inverted, the re-

sulting PMNS matrix always leads to $\sin^2 \theta_{13} = 1/3$, which is much larger than the measured value. Hence in the framework of the effective model, this case is not viable.

Finally, we note that the residual family symmetry in the neutrino sector is $G_\nu = Z_2^S \equiv \{1, S\}$ for all the four cases discussed. This is the reason why the second column of the tri-bimaximal mixing matrix is kept. In summary, in the effective model there are only two viable cases, namely (i) with no CP violation, and (iii) with maximal CP violation $\delta_{CP} = \pm\pi/2$.

5 A renormalisable $S_4 \rtimes H_{CP}$ model

In the effective model, the LO predictions for the flavour mixing parameters will generally be corrected by higher dimensional operators allowed by the imposed symmetries. As a result, the purely effective formulation will leave room for different physical predictions. Moreover, it is generally believed that the non-renormalisable operators in the effective theory should come from a more fundamental underlying renormalisable theory by integrating out the heavy degrees of freedom. Therefore, the improved renormalisable model of this section addresses these questions while giving rise to tri-bimaximal neutrino mixing at leading order which is broken to trimaximal mixing by next to leading order (NLO) contributions. This will naturally explain why the reactor angle and the deviations from maximal atmospheric mixing are relatively small.

The model of this section is inspired by the renormalisable trimaximal S_4 model of [22] which was originally proposed without H_{CP} . Here we shall construct an analogous renormalisable S_4 model but with generalised CP symmetry based on

$$S_4 \rtimes H_{CP} , \quad (5.1)$$

supplemented by the same extra symmetries as in the previous effective model, i.e.

$$Z_4 \times Z_3 \times U(1)_R , \quad (5.2)$$

in order to control the allowed operators in the model. The matter fields, flavon fields and their transformation properties under the imposed symmetries are presented in Table 4, while the driving fields, messenger fields, and their transformation properties are given in Table 5. Tables 4 and 5 may be compared to Table 2 of the effective model.

As before we impose a generalised CP symmetry H_{CP} consistently with S_4 . In addition to forcing the couplings in the superpotential to be real, the generalised CP symmetry will be spontaneously broken. The remaining symmetry in the neutrino sector,

$$G_{CP}^\nu \cong Z_2^S \times H_{CP}^\nu, \quad (5.3)$$

will result in predictions for δ_{CP} and other relations, as discussed in the previous section. However in order to justify this breaking, we must perform a detailed analysis of the flavon potential of the renormalisable model.

Field	L	N^c	e^c	μ^c	τ^c	$H_{u,d}$	φ_T	η	φ_S	ϕ	ξ	Δ
S_4	3	3	1	1'	1	1	3	2	3'	2	1	1'
Z_4	1	1	i	-1	$-i$	1	i	i	1	1	1	1
Z_3	ω	ω^2	ω^2	1	ω	1	ω	ω	ω^2	ω^2	ω^2	1
$U(1)_R$	1	1	1	1	1	0	0	0	0	0	0	0

Table 4: The transformation properties of the matter fields, Higgs and flavon fields in the renormalisable model.

5.1 Vacuum alignment

The renormalisable driving superpotential that is linear in the driving fields and invariant under the flavour symmetry is

$$\begin{aligned}
w_d = & g_1 (\varphi_T^0 (\varphi_T \varphi_T)_{\mathbf{3}'})_{\mathbf{1}} + g_2 (\varphi_T^0 (\eta \varphi_T)_{\mathbf{3}'})_{\mathbf{1}} + g_3 \zeta^0 (\varphi_T \varphi_T)_{\mathbf{1}} + g_4 \zeta^0 (\eta \eta)_{\mathbf{1}} \\
& + f_1 (\varphi_S^0 (\varphi_S \varphi_S)_{\mathbf{3}'})_{\mathbf{1}} + f_2 (\varphi_S^0 (\phi \varphi_S)_{\mathbf{3}'})_{\mathbf{1}} + f_3 (\varphi_S^0 \varphi_S)_{\mathbf{1}} \xi + f_4 (\tilde{\varphi}_S^0 (\phi \varphi_S)_{\mathbf{3}})_{\mathbf{1}} \\
& + f_5 \xi^0 (\varphi_S \varphi_S)_{\mathbf{1}} + f_6 \xi^0 (\phi \phi)_{\mathbf{1}} + f_7 \xi^0 \xi^2 + M^2 \Delta^0 + f_8 \Delta^0 \Delta^2 ,
\end{aligned} \tag{5.4}$$

where the term $\Delta^0 H_u H_d$ has been neglected, as it will play no role in the flavon vacuum alignment because the breaking of S_4 flavour symmetry is typically assumed to occur around the GUT scale. Furthermore, the couplings g_i , f_i and M are real parameters due to the generalised CP symmetry.

Similar to the effective model of section 4, the vacuum alignment for the flavons associated with the charged lepton sector is determined by the F -term conditions of φ_T^0 and ζ^0 :

$$\begin{aligned}
\frac{\partial w_d}{\partial \varphi_{T_1}^0} &= 2g_1(\varphi_{T_1}^2 - \varphi_{T_2} \varphi_{T_3}) + g_2(\eta_1 \varphi_{T_2} - \eta_2 \varphi_{T_3}) = 0 , \\
\frac{\partial w_d}{\partial \varphi_{T_2}^0} &= 2g_1(\varphi_{T_2}^2 - \varphi_{T_1} \varphi_{T_3}) + g_2(\eta_1 \varphi_{T_1} - \eta_2 \varphi_{T_2}) = 0 , \\
\frac{\partial w_d}{\partial \varphi_{T_3}^0} &= 2g_1(\varphi_{T_3}^2 - \varphi_{T_1} \varphi_{T_2}) + g_2(\eta_1 \varphi_{T_3} - \eta_2 \varphi_{T_1}) = 0 , \\
\frac{\partial w_d}{\partial \zeta^0} &= g_3(\varphi_{T_1}^2 + 2\varphi_{T_2} \varphi_{T_3}) + 2g_4 \eta_1 \eta_2 = 0 .
\end{aligned}$$

This set of equations is satisfied by two solutions, where the ambiguity caused by S_4 symmetry transformations is ignored. The first solution is given by

$$\langle \varphi_T \rangle = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} v_T, \quad \langle \eta \rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix} v_\eta, \quad \text{with} \quad v_T^2 = -\frac{2g_4}{3g_3} v_\eta^2, \tag{5.5}$$

while the second is given by

$$\langle \varphi_T \rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} v_T, \quad \langle \eta \rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} v_\eta, \quad \text{with} \quad v_T = \frac{g_2}{2g_1} v_\eta. \tag{5.6}$$

In this work, we choose the second solution since the first solution can be removed by introducing another driving field that transforms identically with ζ^0 , similar to the effective model. Notice that the phase of v_η can be absorbed into the lepton fields. Therefore, we take v_η to be real, implying that the VEV v_T is real as well.

We continue the analysis of the vacuum alignment by considering the flavon fields associated with the neutrino sector, i.e. φ_S , ϕ and ξ . The F -term conditions determining their alignments are

$$\begin{aligned}
\frac{\partial w_d}{\partial \varphi_{S_1}^0} &= 2f_1(\varphi_{S_1}^2 - \varphi_{S_2}\varphi_{S_3}) + f_2(\phi_1\varphi_{S_2} + \phi_2\varphi_{S_3}) + f_3\xi\varphi_{S_1} = 0 , \\
\frac{\partial w_d}{\partial \varphi_{S_2}^0} &= 2f_1(\varphi_{S_2}^2 - \varphi_{S_1}\varphi_{S_3}) + f_2(\phi_1\varphi_{S_1} + \phi_2\varphi_{S_2}) + f_3\xi\varphi_{S_3} = 0 , \\
\frac{\partial w_d}{\partial \varphi_{S_3}^0} &= 2f_1(\varphi_{S_3}^2 - \varphi_{S_1}\varphi_{S_2}) + f_2(\phi_1\varphi_{S_3} + \phi_2\varphi_{S_1}) + f_3\xi\varphi_{S_2} = 0 , \\
\frac{\partial w_d}{\partial \tilde{\varphi}_{S_1}^0} &= f_4(\phi_1\varphi_{S_2} - \phi_2\varphi_{S_3}) = 0 , \\
\frac{\partial w_d}{\partial \tilde{\varphi}_{S_2}^0} &= f_4(\phi_1\varphi_{S_1} - \phi_2\varphi_{S_2}) = 0 , \\
\frac{\partial w_d}{\partial \tilde{\varphi}_{S_3}^0} &= f_4(\phi_1\varphi_{S_3} - \phi_2\varphi_{S_1}) = 0 , \\
\frac{\partial w_d}{\partial \xi^0} &= f_5(\varphi_{S_1}^2 + 2\varphi_{S_2}\varphi_{S_3}) + 2f_6\phi_1\phi_2 + f_7\xi^2 = 0 .
\end{aligned} \tag{5.7}$$

This set of equations leads to vacuum alignments given by

$$\langle \varphi_S \rangle = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} v_S, \quad \langle \phi \rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix} v_\phi, \quad \langle \xi \rangle = u , \tag{5.8}$$

where

$$v_S^2 = -\frac{1}{6f_2^2 f_5} (f_3^2 f_6 + 2f_2^2 f_7) u^2, \quad v_\phi = -\frac{f_3}{2f_2} u . \tag{5.9}$$

Notice that the vacuum alignments of the flavons shown in Eq. (5.8) are invariant under the action of both the S and U elements of S_4 , preserving the tri-bimaximal Klein four subgroup. Furthermore, the phase of u can be absorbed by field redefinition. Hence, u can be taken to be real without loss of generality. This renders v_ϕ real as well, but the VEV v_S can be real or purely imaginary depending on the coefficient $-(f_3^2 f_6 + 2f_2^2 f_7)/(f_2^2 f_5)$ being positive or negative, respectively.

We conclude the discussion of the vacuum alignment of the renormalisable model by considering the last two operators of Eq. (5.4) which are responsible for the alignment of the flavon Δ . They provide the F -term minimisation condition

$$\frac{\partial w_d}{\partial \Delta^0} = M^2 + f_8 \Delta^2 = 0 , \tag{5.10}$$

Field	φ_T^0	ζ^0	φ_S^0	$\tilde{\varphi}_S^0$	ξ^0	Δ^0	Ω_1	Ω_1^c	Ω_2	Ω_2^c	Ω_3	Ω_3^c	Σ	Σ^c
S_4	3'	1	3'	3	1	1	2	2	2	2	3	3	3'	3'
Z_4	-1	-1	1	1	1	1	-1	-1	-i	i	1	1	1	1
Z_3	ω	ω	ω^2	ω^2	ω^2	1	1	1	ω	ω^2	ω^2	ω	ω^2	ω
$U(1)_R$	2	2	2	2	2	2	1	1	1	1	1	1	1	1

Table 5: The driving fields, messenger fields and their transformation rules under the $S_4 \times Z_4 \times Z_3$ and $U(1)_R$ symmetries.

which is satisfied when

$$v_\Delta^2 = -M^2/f_8, \quad (5.11)$$

where $v_\Delta = \langle \Delta \rangle$. Notice that if $f_8 > 0$, then v_Δ is purely imaginary. However if $f_8 < 0$, then v_Δ is completely real. Hence, v_Δ is constrained to be either real or purely imaginary in this model. Thus, having completed the discussion of the vacuum alignment of the renormalisable model, we now proceed to investigate the leptonic masses and mixings predicted by it.

5.2 The structure of the model

The charged lepton sector is formulated at the renormalisable level with the introduction of three pairs of messengers Ω_i and Ω_i^c ($i = 1, 2, 3$). Note that these messengers are chiral superfields with non-vanishing hypercharge $+2(-2)$ for Ω_i (Ω_i^c). With the particles and their transformation properties listed in Tables 4 and 5, we obtain the renormalisable superpotential for the charged leptons,

$$\begin{aligned}
w_\ell = & z_1 (L\Omega_3)_1 H_d + z_2 (\Omega_3^c \varphi_T)_1 \tau^c + z_3 ((\Omega_3^c \varphi_T)_2 \Omega_2)_1 + z_4 (\Omega_2^c \eta)_1 \mu^c \\
& + z_5 ((\Omega_2^c \eta)_2 \Omega_1)_1 + z_6 (\Omega_1^c \eta)_1 e^c + M_{\Omega_1} (\Omega_1 \Omega_1^c)_1 + z_7 \Delta (\Omega_1 \Omega_1^c)_1 \\
& + M_{\Omega_2} (\Omega_2 \Omega_2^c)_1 + z_8 \Delta (\Omega_2 \Omega_2^c)_1 + M_{\Omega_3} (\Omega_3 \Omega_3^c)_1,
\end{aligned} \quad (5.12)$$

where general CP invariance again implies that all the order one coupling constants z_i and the messenger masses M_{Ω_1} , M_{Ω_2} and M_{Ω_3} are real. Furthermore, since the terms $\Delta (\Omega_1 \Omega_1^c)_1$ and $\Delta (\Omega_2 \Omega_2^c)_1$ lead to corrections to the Ω_1 and Ω_2 masses, respectively, and the mass scales of the messenger fields are much larger than the VEVs of the flavons, the contributions of these two operators can be safely neglected.

Integrating out the messenger pairs Ω_i and Ω_i^c (the corresponding Feynman diagrams are shown in Fig. 1), yields the following effective superpotential for the charged lepton masses:

$$\begin{aligned}
w_\ell^{eff} = & -\frac{z_1 z_2}{M_{\Omega_3}} (L\varphi_T)_1 H_d \tau^c + \frac{z_1 z_3 z_4}{M_{\Omega_2} M_{\Omega_3}} ((L\varphi_T)_2 \eta)_1 \mu^c \\
& -\frac{z_1 z_3 z_5 z_6}{M_{\Omega_1} M_{\Omega_2} M_{\Omega_3}} ((L\varphi_T)_2 (\eta\eta)_2)_1 H_d e^c.
\end{aligned}$$

Then, by applying the VEVs in Eq. (5.6), a diagonal charged lepton mass matrix is

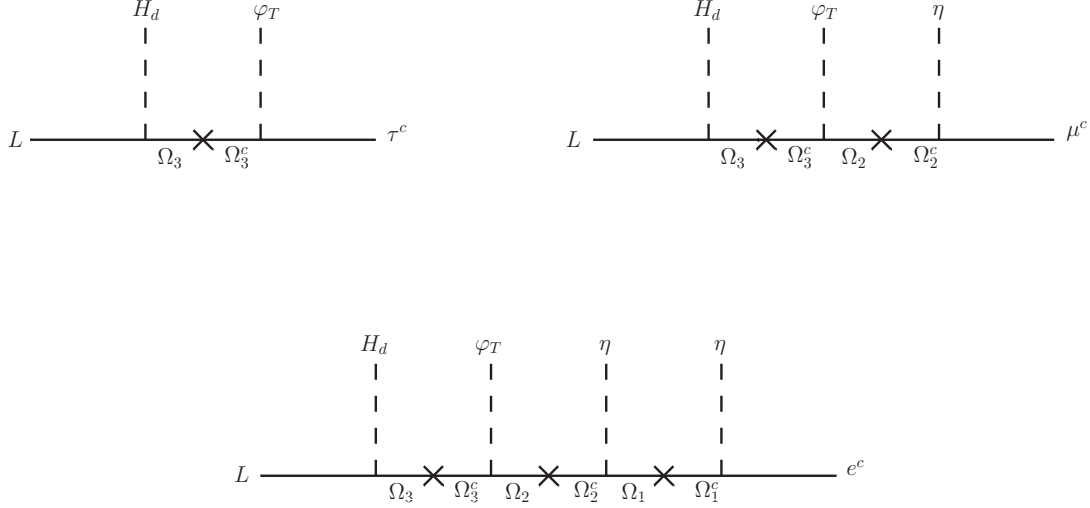


Figure 1: The diagrams which generate the effective operators for the charged lepton masses, where crosses indicate the mass insertions for fermions.

obtained with

$$m_\tau = -z_1 z_2 \frac{v_T}{M_{\Omega_3}} v_d, \quad m_\mu = z_1 z_3 z_4 \frac{v_T v_\eta}{M_{\Omega_2} M_{\Omega_3}} v_d, \quad m_e = -z_1 z_3 z_5 z_6 \frac{v_T v_\eta^2}{M_{\Omega_1} M_{\Omega_2} M_{\Omega_3}} v_d. \quad (5.13)$$

The mass hierarchies among the charged leptons are reproduced naturally without invoking another mechanisms. Here the VEVs of the flavons φ_T and η are responsible for the spontaneous breaking of both flavour and generalised CP symmetries. As the effective model in section 4, the S_4 flavour symmetry is broken completely and the remnant CP symmetry is $H_{CP}^\ell = \rho_{\mathbf{r}}(1)$ in the charged lepton sector.

Having completed the analysis of the charged lepton sector, we now turn to discuss the neutrino sector. We begin this by writing the renormalisable superpotential responsible for the light neutrino masses, which consists of the LO and relevant messenger terms:

$$w_\nu = w_\nu^{LO} + w_\nu^\Sigma, \quad (5.14)$$

where

$$\begin{aligned} w_\nu^{LO} &= y (LN^c)_1 H_u + y_1 ((N^c N^c)_{\mathbf{3}'} \varphi_S)_1 + y_2 (N^c N^c)_1 \xi + y_3 ((N^c N^c)_2 \phi)_1, \\ w_\nu^\Sigma &= x_1 ((N^c \Sigma)_{\mathbf{3}'} \varphi_S)_1 + x_2 ((N^c \Sigma)_2 \phi)_1 + x_3 (N^c \Sigma^c)_1 \Delta + M_\Sigma (\Sigma \Sigma^c)_1. \end{aligned} \quad (5.15)$$

where the messenger field Σ (Σ^c) is a chiral superfield carrying zero hypercharge, and all the parameters x_i and y_i are real due to generalised CP invariance. It is clear that the Dirac neutrino mass matrix takes the simple form

$$m_D = y v_u \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (5.16)$$

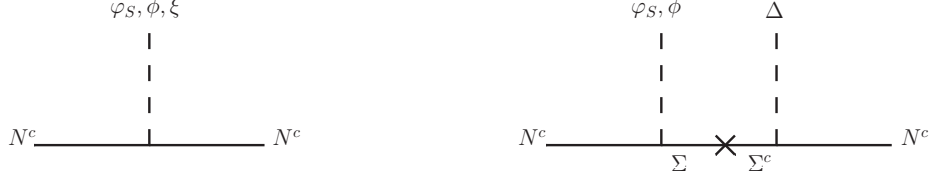


Figure 2: The diagrams for the RH neutrino masses, where the cross indicates a fermionic mass insertion.

Further notice that w_ν^{LO} gives rise to the RH Majorana neutrino mass matrix m_M^{LO} , and after inserting the flavon VEVs of φ_S , ϕ and ξ from Eq. (5.8), this is revealed to be

$$m_M^{LO} = y_1 v_s \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} + y_2 u \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + y_3 v_\phi \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}. \quad (5.17)$$

The resulting effective light neutrino mass matrix $m_\nu^{LO} = -m_D(m_M^{LO})^{-1}m_D^T$ is exactly diagonalised by the tri-bimaximal mixing matrix U_{TB} ,

$$U_{TB}^T m_\nu^{LO} U_{TB} = \text{diag}(m_1^{LO}, m_2^{LO}, m_3^{LO}), \quad (5.18)$$

where the light neutrino masses $m_{1,2,3}^{LO}$ are

$$m_1^{LO} = -\frac{y^2 v_u^2}{3y_1 v_s + y_2 u - y_3 v_\phi}, \quad m_2^{LO} = -\frac{y^2 v_u^2}{y_2 u + 2y_3 v_\phi}, \quad m_3^{LO} = -\frac{y^2 v_u^2}{3y_1 v_s - y_2 u + y_3 v_\phi}. \quad (5.19)$$

The reason why the tri-bimaximal mixing is produced is because the VEVs of φ_S , ϕ and ξ preserve the Klein four subgroup generated by the tri-bimaximal S and U generators, as has been pointed out in section 5.1. The LO and NLO contributions to the RH neutrino masses are shown diagrammatically in Fig. 2. Integrating out the messenger fields Σ and Σ^c yields the NLO effective operator

$$w_\nu^{NLO} = -\frac{x_2 x_3}{M_\Sigma} \Delta((N^c N^c)_2 \phi)_{1'}. \quad .$$

The effective operator $\Delta((N^c N^c)_3 \varphi_S)_{1'}$, is not reproduced, since the contraction $N^c N^c$ vanishes from the antisymmetry of the associated S_4 Clebsch-Gordan coefficients, as is shown in Appendix B. Straightforwardly, we see that w_ν^{NLO} gives rise to a NLO contribution to m_M of the form

$$m_M^{NLO} = x_2 x_3 \frac{v_\Delta v_\phi}{M_\Sigma} \left[\begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \right].$$

Note that the NLO contribution m_M^{NLO} is induced by the flavon Δ which further breaks the remnant Klein four symmetry to Z_2^S . Then, the RH neutrino mass matrix m_M including

the NLO contribution can be expressed as

$$\begin{aligned}
m_M &= m_M^{LO} + m_M^{NLO} \\
&= \tilde{a} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} + \tilde{b} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \tilde{c} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \tilde{d} \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad (5.20)
\end{aligned}$$

where the parameters \tilde{a} , \tilde{b} , \tilde{c} and \tilde{d} are defined as

$$\tilde{a} = y_1 v_S, \quad \tilde{b} = y_2 u, \quad \tilde{c} = y_3 v_\phi, \quad \tilde{d} = x_2 x_3 \frac{v_\Delta v_\phi}{M_\Sigma}. \quad (5.21)$$

The light neutrino mass matrix m_ν is given by the seesaw formula

$$\begin{aligned}
m_\nu &= -m_D m_M^{-1} m_D^T \\
&= \alpha \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} + \beta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}. \quad (5.22)
\end{aligned}$$

It is the most general neutrino mass matrix consistent with the residual Z_2^S flavour symmetry, as is shown in Eq. (3.12). The parameters α , β , γ and ϵ are given by,

$$\begin{aligned}
\alpha &= \frac{\tilde{a}}{-9\tilde{a}^2 + (\tilde{b} - \tilde{c})^2 + 3\tilde{d}^2}, \quad \beta = -\frac{1}{3(\tilde{b} + 2\tilde{c})} + \frac{2(\tilde{b} - \tilde{c})}{3[9\tilde{a}^2 - (\tilde{b} - \tilde{c})^2 - 3\tilde{d}^2]}, \\
\gamma &= -\frac{1}{3(\tilde{b} + 2\tilde{c})} - \frac{\tilde{b} - \tilde{c}}{3[9\tilde{a}^2 - (\tilde{b} - \tilde{c})^2 - 3\tilde{d}^2]}, \quad \epsilon = \frac{\tilde{d}}{-9\tilde{a}^2 + (\tilde{b} - \tilde{c})^2 + 3\tilde{d}^2}, \quad (5.23)
\end{aligned}$$

where the overall factor $y^2 v_u^2$ has been omitted. We note that the first three terms in the light neutrino matrix of Eq. (5.22) preserve tri-bimaximal mixing while the last term, proportional to ϵ , violates it. In the present model, the ϵ term is induced by the NLO contributions and hence suppressed by v_Δ/M_Σ with respect to α , β and γ . This provides a natural explanation as to why the reactor angle as well as the deviations from maximal atmospheric mixing are relatively small. Although, their definite values cannot be predicted.

From the vacuum alignment in section 5.1, we know that both \tilde{b} and \tilde{c} are real parameters, and \tilde{a} and \tilde{d} can be real or purely imaginary. Therefore, there are four possible cases allowed by the domains of the parameters in the renormalisable model, these are summarised in Table 6. It turns out that each of these four cases will preserve different generalised CP transformations as well as lead to different phenomenological predictions. In what follows, we shall discuss the four cases in Table 6, one by one.

- (i) In this case, both the triplet VEV v_S and the singlet VEV v_Δ are real. The generalised CP symmetries for $g = 1$ and $g = S$ are preserved in the neutrino sector, i.e. the residual CP symmetry in the neutrino sector is $H_{CP}' = \{\rho_{\mathbf{r}}(1), \rho_{\mathbf{r}}(S)\}$, where \mathbf{r}

		α	β	γ	ϵ
(i)	$\tilde{a} \in \mathbb{R}, \tilde{b} \in \mathbb{R}, \tilde{c} \in \mathbb{R}, \tilde{d} \in \mathbb{R}$	real	real	real	real
(ii)	$\tilde{a} \in \mathbb{C}, \tilde{b} \in \mathbb{R}, \tilde{c} \in \mathbb{R}, \tilde{d} \in \mathbb{R}$ with $\tilde{a} = -\tilde{a}^*$	imaginary	real	real	real
(iii)	$\tilde{a} \in \mathbb{R}, \tilde{b} \in \mathbb{R}, \tilde{c} \in \mathbb{R}, \tilde{d} \in \mathbb{C}$ with $\tilde{d} = -\tilde{d}^*$	real	real	real	imaginary
(iv)	$\tilde{a} \in \mathbb{C}, \tilde{b} \in \mathbb{R}, \tilde{c} \in \mathbb{R}, \tilde{d} \in \mathbb{C}$ with $\tilde{a} = -\tilde{a}^*$ and $\tilde{d} = -\tilde{d}^*$	imaginary	real	real	imaginary

Table 6: The four allowed ranges of α , β , γ and ϵ as dictated by the domains of \tilde{a} , \tilde{b} , \tilde{c} and \tilde{d} in the renormalisable model.

denotes the irreducible representations of S_4 . For the corresponding light neutrino mass matrix shown in Eq. (5.22), the parameters α , β , γ and ϵ are all real, yielding a matrix that is precisely the same as the neutrino mass matrix of case (i) from the general analysis in section 3. Therefore, the neutrino mixing matrix is of trimaximal form, and the predictions for lepton mixing angles and CP phases are given by Eqs. (3.22,3.23). There is no CP violation, as the CP symmetry $X_{\mathbf{r}} = \rho_{\mathbf{r}}(1)$ is conserved in both the neutrino and charged lepton sectors. However this case gives a viable description of the lepton mixing angles.

- (ii) In this case, v_S is purely imaginary and v_{Δ} is real. The generalised CP symmetry is broken to $H_{CP}^{\nu} = \{\rho_{\mathbf{r}}(TST^2), \rho_{\mathbf{r}}(T^2ST)\}$ in the neutrino sector. The resulting parameter α is imaginary and β , γ and ϵ are real. As a result, this is case (ii) discussed in section 3, and the light neutrino masses are degenerate, i.e. $|m_1| = |m_3|$. Hence, this case is not viable.
- (iii) This case corresponds to the VEV v_S being real and v_{Δ} being purely imaginary. Only two of the 24 generalised CP symmetries are preserved in the neutrino sector, i.e. $H_{CP}^{\nu} = \{\rho_{\mathbf{r}}(U), \rho_{\mathbf{r}}(SU)\}$. Regarding the light neutrino mass matrix of Eq. (5.22), the tri-bimaximal violating parameter ϵ is imaginary and α , β and γ are real. Therefore, this case is identical to case (iii) of the general analysis inspired by symmetry arguments, and the predictions for its mixing angles and CP phases are as given in Eqs. (3.28,3.29). Notice that this case produces maximal Dirac CP violation $|\delta_{CP}| = \pi/2$, which is the result of the mismatch between the remnant CP symmetries $H_{CP}^{\ell} = \{\rho_{\mathbf{r}}(1)\}$ and $H_{CP}^{\nu} = \{\rho_{\mathbf{r}}(U), \rho_{\mathbf{r}}(SU)\}$. This case is therefore viable.
- (iv) Both the VEVs v_S and v_{Δ} are purely imaginary in this case, the remnant CP symmetry in the neutrino sector turns out to be $H_{CP}^{\nu} = \{\rho_{\mathbf{r}}(TST^2U), \rho_{\mathbf{r}}(T^2STU)\}$. The parameters α and ϵ are imaginary while β and γ are real. Hence, the corresponding light neutrino mass matrix has the same form as that of case (iv) analysed in section 3, and the resulting PMNS matrix is given in Eq. (3.33) up to permutations of rows and columns. However, in the present model, the mass hierarchies among the three charged lepton are reproduced, as is shown in Eq. (5.13). Consequently, permuting the rows of the PMNS matrix is forbidden. Notice that even if we desired to exchange its columns, as the neutrino mass order is less constrained so far, we would

always get $\sin^2 \theta_{13} = 1/3$, which is much larger than the experimental observations. Therefore, we conclude that this case is not viable for the renormalisable model.

In summary, in the renormalisable model there are only two viable cases, namely (i) with no CP violation, and (iii) with maximal CP violation. This is the same as in the effective model.

6 Conclusions

The measurement of the reactor mixing angle, which is observed to be rather large, offers some encouragement that the measurement of leptonic CP violation, in particular the Dirac oscillation phase δ_{CP} , may be possible in the not too distant future. This has led to renewed interest in theories that are able to predict the value of δ_{CP} .

In this work, we have focused on a promising framework where (generalised) CP symmetry in the lepton sector is combined with a discrete family symmetry (and perhaps other symmetries). The CP and family symmetries are subsequently spontaneously broken, with different subgroups preserved in the neutrino and charged lepton sectors. The models presented in this paper provide a generalisation for the direct models reviewed in [8] to include CP symmetry.

Within this framework, we have addressed the question of spontaneous breaking of a generalised CP symmetry in particular for models based on an S_4 family symmetry. We have constructed two models of leptons based on S_4 family symmetry combined with a generalised CP symmetry H_{CP} , one at the effective level and another one at the renormalisable level. In both models, we have shown how the flavon potential can spontaneously break the symmetry $S_4 \rtimes H_{CP}$ down to $Z_2 \times H_{CP}^\nu$ in the neutrino sector. This symmetry breaking was simply assumed to happen in [20] without any dynamical justification.

In our models, the choice of preserved CP symmetry H_{CP}^ν in the neutrino sector is controlled by free (real) parameters in the flavon potential, enabling us to dial the type of CP violation. Of the two realistic models of this kind that we have proposed, one at the effective level and another one at the renormalisable level, we find that both models predict trimaximal lepton mixing with CP being either fully preserved or maximally broken and the intermediate possibility forbidden by the structure of the models.

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Appendices

A Implications of a generalised CP symmetry

It is well-known that at low energies there are three CP violating phases in the lepton sector: one of the Dirac type and two of the Majorana type. The strength of leptonic CP violation of the Dirac type, which can be observable through neutrino oscillations, is determined by the following CP-odd weak basis (WB) invariant [24]:

$$\mathcal{J}_1 \equiv \text{Tr} [h_\nu^*, h_l]^3, \quad \text{with} \quad h_\nu = m_\nu m_\nu^\dagger \quad \text{and} \quad h_l = m_l m_l^\dagger. \quad (\text{A.1})$$

It can be fully written in terms of physical observables as

$$\mathcal{J}_1 = -6i (m_\mu^2 - m_e^2) (m_\tau^2 - m_\mu^2) (m_\tau^2 - m_e^2) \Delta m_{21}^2 \Delta m_{31}^2 \Delta m_{32}^2 J_{CP}, \quad (\text{A.2})$$

where $\Delta m_{ij}^2 = m_i^2 - m_j^2$ are the light neutrino mass squared differences and the quantity J_{CP} is the Jarlskog invariant [25],

$$\begin{aligned} J_{CP} &= \text{Im} [(U_{PMNS})_{11} (U_{PMNS})_{22} (U_{PMNS}^*)_{12} (U_{PMNS}^*)_{21}] \\ &= \frac{1}{8} \sin(2\theta_{23}) \sin(2\theta_{12}) \sin(2\theta_{13}) \cos \theta_{13} \sin \delta_{CP}. \end{aligned} \quad (\text{A.3})$$

where the PMNS matrix U_{PMNS} has been parameterised as in Eq. (3.17), θ_{ij} are the mixing angles chosen to lie in the first quadrant, and δ_{CP} is the Dirac CP-violating phase. It is obvious that \mathcal{J}_1 vanishes for $\delta_{CP} = 0$, and vice versa the Dirac phase δ_{CP} would be zero if \mathcal{J}_1 vanishes. From Eq. (2.3), we see that the generalised CP invariance implies

$$h_l = X_L h_l^T X_L^\dagger, \quad h_\nu = X_L^* h_\nu^* X_L^T. \quad (\text{A.4})$$

Therefore, we have

$$\mathcal{J}_1 = \text{Tr} [h_\nu^*, h_l]^3 = \text{Tr} (X_L [h_\nu, h_l^T]^3 X_L^\dagger) = \text{Tr} [(h_\nu^*)^T, h_l^T]^3 = -\text{Tr} [h_\nu^*, h_l]^3 = 0. \quad (\text{A.5})$$

This indicates that there is no Dirac type CP violation if the generalised CP symmetry is preserved. In the case of Majorana neutrinos, there is also the possibility of the Majorana type CP violation. It has been established that the vanishing of the WB invariant \mathcal{J}_1 together with the following two WB invariants,

$$\begin{aligned} \mathcal{J}_2 &= \text{Im} \text{Tr} (h_l h_\nu^* m_\nu^* h_l^* m_\nu) , \\ \mathcal{J}_3 &= \text{Tr} [m_\nu h_l m_\nu^*, h_l^*]^3 , \end{aligned} \quad (\text{A.6})$$

provides necessary and sufficient conditions for low energy CP invariance [21, 26–28]. Notice that \mathcal{J}_2 and \mathcal{J}_3 measure the Majorana type CP violation. Then, from the generalised CP invariance requirement of Eq. (2.3), we have

$$\mathcal{J}_2 = \text{Im} \text{Tr} (X_L h_l^* h_\nu m_\nu h_l m_\nu^* X_L^\dagger) = \text{Im} \text{Tr} (h_l h_\nu^* m_\nu^* h_l^* m_\nu)^* = 0, \quad (\text{A.7})$$

$$\mathcal{J}_3 = \text{Tr} (X_L^* [m_\nu^* h_l^* m_\nu, h_l]^3 X_L^T) = \text{Tr} [(m_\nu h_l m_\nu^*)^T, (h_l^*)^T]^3 = -\text{Tr} [m_\nu h_l m_\nu^*, h_l^*]^3 = 0.$$

This means that CP will be conserved if the theory is invariant under a generalised CP transformation. As a result, the generalised CP symmetry must be spontaneously or explicitly broken in order to generate non-trivial CP phases.

	S	T	U
1, 1'	1	1	± 1
2	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
3, 3'	$\frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}$	$\mp \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Table 7: The S_4 representation matrices for the S , T and U elements in different irreducible representations, where $\omega = e^{2\pi i/3}$.

B Group theory of S_4

S_4 is the permutation group of four distinct objects, and it is isomorphic to the symmetry group of a regular octahedron. It has $4! = 24$ elements and can be expressed in terms of two generators. However, in order to clearly see the connection to the groups A_4 and S_3 it is useful to express S_4 in terms of three generators, S , T and U [29], which obey the multiplication rules

$$S^2 = T^3 = U^2 = (ST)^3 = (SU)^2 = (TU)^2 = (STU)^4 = 1 . \quad (\text{B.1})$$

Notice that the generators S and T alone generate the group A_4 , while the generators T and U alone generate the group S_3 . Taking all possible combinations of S , T , and U (subject to the rules of Eq. (B.1)), yields the 24 elements of S_4 which belong to 5 disjoint conjugacy classes. To emphasise the geometric aspect of S_4 , we adopt Schoenflies notation, in which kC_n designates a conjugacy class of k elements that are all rotations by $\frac{2\pi}{n}$, to express the conjugacy classes of S_4 as

$$\begin{aligned}
1C_1 &= \{1\} , \\
3C_2 &= \{S, TST^2, T^2ST\} , \\
6C'_2 &= \{U, TU, SU, T^2U, STSU, ST^2SU\} , \\
8C_3 &= \{T, ST, TS, STS, T^2, ST^2, T^2S, ST^2S\} , \\
6C_4 &= \{STU, TSU, T^2SU, ST^2U, TST^2U, T^2STU\} .
\end{aligned} \quad (\text{B.2})$$

These conjugacy classes can be used to deduce the various irreducible representations of S_4 because (by theorem) the number of the irreducible representations must equal to the number of conjugacy classes. Thus, S_4 has five irreducible presentations. Additionally, the sum of the squares of the dimensions of the irreducible representations must equal the order of the group, *i.e.* 24. This implies the five irreducible representations of S_4 are two 1-dimensional (**1**, **1'**), one 2-dimensional (**2**), and two 3-dimensional (**3** and **3'**) irreducible representations. Our choice of the explicit basis for the representation matrices of S , T and U is listed in Table 7 [22]. Furthermore, the Kronecker products of these 5 different

irreducible representations are

$$\begin{aligned}
\mathbf{1} \otimes \mathbf{R} &= \mathbf{R} , \quad \mathbf{1}' \otimes \mathbf{1}' = \mathbf{1} , \quad \mathbf{1}' \otimes \mathbf{2} = \mathbf{2} , \quad \mathbf{1}' \otimes \mathbf{3} = \mathbf{3}' , \quad \mathbf{1}' \otimes \mathbf{3}' = \mathbf{3} , \\
\mathbf{2} \otimes \mathbf{2} &= (\mathbf{1} \oplus \mathbf{2})_s \oplus \mathbf{1}'_a , \quad \mathbf{2} \otimes \mathbf{3} = \mathbf{2} \otimes \mathbf{3}' = \mathbf{3} \oplus \mathbf{3}' , \\
\mathbf{3} \otimes \mathbf{3} &= \mathbf{3}' \otimes \mathbf{3}' = (\mathbf{1} \oplus \mathbf{2} \oplus \mathbf{3}')_s \oplus \mathbf{3}_a , \quad \mathbf{3} \otimes \mathbf{3}' = \mathbf{1}' \oplus \mathbf{2} \oplus \mathbf{3} \oplus \mathbf{3}' ,
\end{aligned} \tag{B.3}$$

where \mathbf{R} stands for any S_4 representation, and the index s (a) denotes symmetric (anti-symmetric) combinations. These Kronecker products, along with the explicit forms of the generators in Table 7 can be used to calculate the corresponding Clebsch-Gordan (CG) coefficients. These CG coefficients can be also be found in other works, e.g. Ref. [22], but we list them here for completeness. In the following reporting of the CG coefficients of S_4 , we use α_i to denote the elements of the first representation and β_j to indicate those of the second representation of the product. Furthermore, “ n ” counts the number of “primes” in the Kronecker product (e.g. in $\mathbf{1} \otimes \mathbf{1}' = \mathbf{1}'$, $n = 2$).

$$\begin{aligned}
\mathbf{1}^{(n)} \otimes \mathbf{1}^{(n)} &\rightarrow \mathbf{1}^{(n)} \left\{ \begin{array}{l} n = \text{even} \\ n = \text{odd} \end{array} \right. \left\{ \begin{array}{l} \mathbf{1} \otimes \mathbf{1} \rightarrow \mathbf{1} \\ \mathbf{1}' \otimes \mathbf{1}' \rightarrow \mathbf{1} \\ \mathbf{1} \otimes \mathbf{1}' \rightarrow \mathbf{1}' \end{array} \right\} \alpha\beta , \\
\mathbf{1}^{(n)} \otimes \mathbf{2} &\rightarrow \mathbf{2} \left\{ \begin{array}{l} n = \text{even} \\ n = \text{odd} \end{array} \right. \left\{ \begin{array}{l} \mathbf{1} \otimes \mathbf{2} \rightarrow \mathbf{2} \\ \mathbf{1}' \otimes \mathbf{2} \rightarrow \mathbf{2} \end{array} \right\} \alpha \begin{pmatrix} \beta_1 \\ (-1)^n \beta_2 \end{pmatrix} , \\
\mathbf{1}^{(n)} \otimes \mathbf{3}^{(n)} &\rightarrow \mathbf{3}^{(n)} \left\{ \begin{array}{l} n = \text{even} \end{array} \right. \left\{ \begin{array}{l} \mathbf{1} \otimes \mathbf{3} \rightarrow \mathbf{3} \\ \mathbf{1}' \otimes \mathbf{3}' \rightarrow \mathbf{3} \\ \mathbf{1} \otimes \mathbf{3}' \rightarrow \mathbf{3}' \\ \mathbf{1}' \otimes \mathbf{3} \rightarrow \mathbf{3}' \end{array} \right\} \alpha \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} , \\
\mathbf{2} \otimes \mathbf{2} &\rightarrow \mathbf{1}^{(n)} \left\{ \begin{array}{l} n = \text{even} \\ n = \text{odd} \end{array} \right. \left\{ \begin{array}{l} \mathbf{2} \otimes \mathbf{2} \rightarrow \mathbf{1} \\ \mathbf{2} \otimes \mathbf{2} \rightarrow \mathbf{1}' \end{array} \right\} \alpha_1 \beta_2 + (-1)^n \alpha_2 \beta_1 , \\
\mathbf{2} \otimes \mathbf{2} &\rightarrow \mathbf{2} \left\{ \begin{array}{l} n = \text{even} \end{array} \right. \left\{ \begin{array}{l} \mathbf{2} \otimes \mathbf{2} \rightarrow \mathbf{2} \end{array} \right\} \begin{pmatrix} \alpha_2 \beta_2 \\ \alpha_1 \beta_1 \end{pmatrix} , \\
\mathbf{2} \otimes \mathbf{3}^{(n)} &\rightarrow \mathbf{3}^{(n)} \left\{ \begin{array}{l} n = \text{even} \\ n = \text{odd} \end{array} \right. \left\{ \begin{array}{l} \mathbf{2} \otimes \mathbf{3} \rightarrow \mathbf{3} \\ \mathbf{2} \otimes \mathbf{3}' \rightarrow \mathbf{3}' \\ \mathbf{2} \otimes \mathbf{3} \rightarrow \mathbf{3}' \\ \mathbf{2} \otimes \mathbf{3}' \rightarrow \mathbf{3} \end{array} \right\} \alpha_1 \begin{pmatrix} \beta_2 \\ \beta_3 \\ \beta_1 \end{pmatrix} + (-1)^n \alpha_2 \begin{pmatrix} \beta_3 \\ \beta_1 \\ \beta_2 \end{pmatrix} , \\
\mathbf{3}^{(n)} \otimes \mathbf{3}^{(n)} &\rightarrow \mathbf{1}^{(n)} \left\{ \begin{array}{l} n = \text{even} \end{array} \right. \left\{ \begin{array}{l} \mathbf{3} \otimes \mathbf{3} \rightarrow \mathbf{1} \\ \mathbf{3}' \otimes \mathbf{3}' \rightarrow \mathbf{1} \\ \mathbf{3} \otimes \mathbf{3}' \rightarrow \mathbf{1}' \end{array} \right\} \alpha_1 \beta_1 + \alpha_2 \beta_3 + \alpha_3 \beta_2 , \\
\mathbf{3}^{(n)} \otimes \mathbf{3}^{(n)} &\rightarrow \mathbf{2} \left\{ \begin{array}{l} n = \text{even} \\ n = \text{odd} \end{array} \right. \left\{ \begin{array}{l} \mathbf{3} \otimes \mathbf{3} \rightarrow \mathbf{2} \\ \mathbf{3}' \otimes \mathbf{3}' \rightarrow \mathbf{2} \\ \mathbf{3} \otimes \mathbf{3}' \rightarrow \mathbf{2} \end{array} \right\} \begin{pmatrix} \alpha_2 \beta_2 + \alpha_3 \beta_1 + \alpha_1 \beta_3 \\ (-1)^n (\alpha_3 \beta_3 + \alpha_1 \beta_2 + \alpha_2 \beta_1) \end{pmatrix} ,
\end{aligned}$$

$$\begin{aligned}
\mathbf{3}^{(n)} \otimes \mathbf{3}^{(n)} &\rightarrow \mathbf{3}^{(n)} \left\{ \begin{array}{l} n = \text{odd} \\ \mathbf{3} \otimes \mathbf{3} \rightarrow \mathbf{3}' \\ \mathbf{3} \otimes \mathbf{3}' \rightarrow \mathbf{3} \\ \mathbf{3}' \otimes \mathbf{3}' \rightarrow \mathbf{3}' \end{array} \right\} \begin{pmatrix} 2\alpha_1\beta_1 - \alpha_2\beta_3 - \alpha_3\beta_2 \\ 2\alpha_3\beta_3 - \alpha_1\beta_2 - \alpha_2\beta_1 \\ 2\alpha_2\beta_2 - \alpha_3\beta_1 - \alpha_1\beta_3 \end{pmatrix}, \\
\mathbf{3}^{(n)} \otimes \mathbf{3}^{(n)} &\rightarrow \mathbf{3}^{(n)} \left\{ \begin{array}{l} n = \text{even} \\ \mathbf{3} \otimes \mathbf{3} \rightarrow \mathbf{3} \\ \mathbf{3}' \otimes \mathbf{3}' \rightarrow \mathbf{3} \\ \mathbf{3} \otimes \mathbf{3}' \rightarrow \mathbf{3}' \end{array} \right\} \begin{pmatrix} \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_1\beta_2 - \alpha_2\beta_1 \\ \alpha_3\beta_1 - \alpha_1\beta_3 \end{pmatrix}.
\end{aligned}$$

Now that we have discussed the simpler aspects of the group theory of S_4 , we turn to the more complex topic of the automorphisms of the group S_4 , i.e. the generalised CP transformations $X_{\mathbf{r}}$.

Although the most general solution for the $X_{\mathbf{r}}$ is given by the representation $\rho_{\mathbf{r}}(h)$ of the S_4 group elements $h \in S_4$, as is claimed in Ref. [17], it is still instructive to see how this explicitly occurs. Let us first consider the consistency equation Eq. (2.7) for the generators S , T and U in the faithful representations $\mathbf{3}$ and $\mathbf{3}'$:

$$\begin{aligned}
X_{\mathbf{3}(\mathbf{3}')} \rho_{\mathbf{3}(\mathbf{3}')}^*(S) X_{\mathbf{3}(\mathbf{3}')}^{-1} &= \rho_{\mathbf{3}(\mathbf{3}')} (S'), \\
X_{\mathbf{3}(\mathbf{3}')} \rho_{\mathbf{3}(\mathbf{3}')}^*(T) X_{\mathbf{3}(\mathbf{3}')}^{-1} &= \rho_{\mathbf{3}(\mathbf{3}')} (T'), \\
X_{\mathbf{3}(\mathbf{3}')} \rho_{\mathbf{3}(\mathbf{3}')}^*(U) X_{\mathbf{3}(\mathbf{3}')}^{-1} &= \rho_{\mathbf{3}(\mathbf{3}')} (U').
\end{aligned} \tag{B.4}$$

It is clear from the above equations that the orders of primed and unprimed generators must be identical, i.e. S' and U' must be order 2 elements and T' an order 3 element. As a result, the consistency equations constrain the mappings (automorphisms) such that the generators S, T, U can only be mapped into specific (unions of) conjugacy classes. Namely,

$$S' \in 3C_2 \cup 6C_2', \quad T' \in 8C_3, \quad U' \in 3C_2 \cup 6C_2'. \tag{B.5}$$

Additional constraints can be derived by considering the singlet representation $\mathbf{1}'$. Since $S = T = 1$ and $U = -1$, S' cannot be in the conjugacy class $6C_2'$. Likewise the element U' cannot be in the conjugacy class $3C_2$. This leaves us the automorphisms

$$S' \in 3C_2, \quad T' \in 8C_3, \quad U' \in 6C_2'. \tag{B.6}$$

By investigating all $3 \times 8 \times 6 = 144$ possible values of S' , T' and U' , we find only 24 solutions that satisfy the corresponding consistency equations Eq. (B.4), which can be compactly written as

$$X_{\mathbf{3}(\mathbf{3}')} = \rho_{\mathbf{3}(\mathbf{3}')} (h), \quad h \in S_4, \tag{B.7}$$

with

$$\begin{aligned}
X_{\mathbf{3}(\mathbf{3}')} \rho_{\mathbf{3}(\mathbf{3}')}^*(S) X_{\mathbf{3}(\mathbf{3}')}^{-1} &= \rho_{\mathbf{3}(\mathbf{3}')} (hSh^{-1}), \\
X_{\mathbf{3}(\mathbf{3}')} \rho_{\mathbf{3}(\mathbf{3}')}^*(T) X_{\mathbf{3}(\mathbf{3}')}^{-1} &= \rho_{\mathbf{3}(\mathbf{3}')} (hT^2h^{-1}), \\
X_{\mathbf{3}(\mathbf{3}')} \rho_{\mathbf{3}(\mathbf{3}')}^*(U) X_{\mathbf{3}(\mathbf{3}')}^{-1} &= \rho_{\mathbf{3}(\mathbf{3}')} (hUh^{-1}).
\end{aligned} \tag{B.8}$$

Continuing to the **2**-dimensional irreducible representation, we have

$$\rho_{\mathbf{2}}^*(S) = \rho_{\mathbf{2}}(S), \quad \rho_{\mathbf{2}}^*(T) = \rho_{\mathbf{2}}(T^2), \quad \rho_{\mathbf{2}}^*(U) = \rho_{\mathbf{2}}(U). \tag{B.9}$$

Thus, the generalised CP transformations consistent with the faithful three dimensional irreducible representations are

$$X_{\mathbf{2}} = \rho_2(h), \quad h \in S_4. \quad (\text{B.10})$$

Finally, for the singlet representations $\mathbf{1}$ and $\mathbf{1}'$, we take

$$X_{\mathbf{1}(\mathbf{1}')} = \rho_{\mathbf{1}(\mathbf{1}')} (h), \quad h \in S_4. \quad (\text{B.11})$$

Therefore the generalised CP transformation consistent with an S_4 family symmetry is of the same form as the flavour group transformation, i.e.

$$X_{\mathbf{r}} = \rho_{\mathbf{r}}(h), \quad h \in S_4. \quad (\text{B.12})$$

This provides a proof to the statement in Ref. [17] that the generalised CP transformation group consistent with a S_4 flavour group is S_4 .¹⁰

We end this appendix by noting that our basis for S , T and U is related to the basis choice in Ref. [20] (denoted here as \tilde{S} , \tilde{T} and \tilde{U}) by the unitary transformation

$$V = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{i}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{i}{\sqrt{2}} \end{pmatrix}, \quad (\text{B.13})$$

such that (for the $\mathbf{3}'$ representation)

$$\begin{aligned} \tilde{S} &= V^\dagger S V = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \tilde{T} = V^\dagger T V = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ -1 & \sqrt{2} & -1 \end{pmatrix}, \\ \tilde{U} &= V^\dagger U V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \end{aligned} \quad (\text{B.14})$$

and the generalised CP transformations in the two bases are related by

$$\tilde{X}_{\mathbf{3}(\mathbf{3}')} = V^\dagger X_{\mathbf{3}(\mathbf{3}')} V^*. \quad (\text{B.15})$$

It is interesting to note that the matrix V satisfies

$$V^\dagger V^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \tilde{U}, \quad (\text{B.16})$$

so that $\tilde{X}_{\mathbf{3}(\mathbf{3}')} = V^\dagger X_{\mathbf{3}(\mathbf{3}')} V \tilde{U}$ is explicitly an element of S_4 in the basis with the tilde, given that $X_{\mathbf{3}(\mathbf{3}')}$ is an element of S_4 in our basis.

¹⁰Said again, the generalised CP transformation group is the identity up to inner automorphisms.

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